



A NOTE ON SINGULAR INTEGRALS WITH DOMINATING MIXED SMOOTHNESS IN TRIEBEL-LIZORKIN SPACES*

Hung Viet LE

Faculty of Science and Technology, Hoa Sen University, Quang Trung Software Park,
Section 10, Ward Tan Chanh Hiep, District 12, Ho Chi Minh City, Viet Nam
E-mail: hung.leviet@hoasen.edu.vn; hvle2008@yahoo.com

Abstract Let h be a measurable function defined on $\mathbb{R}^+ \times \mathbb{R}^+$. Let $\Omega \in L(\log L^+)^{\nu_q}(S^{n_1-1} \times S^{n_2-1})$ ($1 \leq \nu_q \leq 2$) be homogeneous of degree zero and satisfy certain cancellation conditions. We show that the singular integral

$$Tf(x_1, x_2) = \text{p. v.} \int_{\mathbb{R}^{n_1+n_2}} \int_{\mathbb{R}^{n_1+n_2}} \frac{\Omega(y'_1, y'_2)h(|y_1|, |y_2|)}{|y_1|^{n_1}|y_2|^{n_2}} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2$$

maps from $S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ boundedly to itself for $1 < p, q < \infty$, $\alpha_1, \alpha_2 \in \mathbb{R}$.

Key words singular integrals; Marcinkiewicz integrals; mixed smoothness; Triebel-Lizorkin spaces

2010 MR Subject Classification 42B20; 42B25

1 Introduction

The subject of singular integral operators has been investigated intensively for several decades. As a consequence, a vast number of interesting results has been discovered by many authors. Recently, the study of L^p -boundedness for singular integrals in product domains has drawn attention to many mathematicians. For a survey of this topic, the reader may view [1–11] among many other excellent references. We remark that the condition that $\Omega \in L(\log L^+)^2(S^{n_1-1} \times S^{n_2-1})$ is optimal for the $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ boundedness of singular integrals in product domains. In fact, it was proved in [2] that $\forall \epsilon > 0$, there is an $\Omega \in L(\log L^+)^{2-\epsilon}(S^{n_1-1} \times S^{n_2-1})$ (satisfying certain cancellation conditions) such that the singular integral

$$T_\Omega f(x_1, x_2) = \text{p. v.} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \frac{\Omega(y'_1, y'_2) f(x_1 - y_1, x_2 - y_2)}{|y_1|^{n_1} |y_2|^{n_2}} dy_1 dy_2$$

is not bounded on $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for any $p \in (1, \infty)$.

On the other hand, consider the Marcinkiewicz integral $\mu_\Omega(f)$ defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_\Omega(x, t)|^2 \frac{dt}{t^3} \right)^{1/2}, \text{ where } F_\Omega(x, t) = \int_{|y| \leq t} \frac{\Omega(y')}{|y|^{n-1}} f(x - y) dy.$$

*Received January 6, 2012; revised April 2, 2013.

In 1972, the author in [15] showed that if $\Omega \in L(\log L^+)^{1/2}(S^{n-1})$ is homogeneous of degree zero and satisfies the cancellation condition, then the Marcinkiewicz integral $\mu_\Omega(f)$ is a bounded operator on $L^2(\mathbb{R}^n)$. In 2002, under the same hypothesis of Ω as in [15], the authors in [3] extended the result in [15] from $p = 2$ to the whole range $1 < p < \infty$. Recently, the authors in [10] generalized the above results to Triebel-Lizorkin product spaces as follows.

Theorem 1.1 [10] Let $2 < q \leq \infty$, $1 < p < \infty$. If $\Omega \in L(\log L^+)^{2/q}(S^{m-1} \times S^{n-1})$, then

$$\|\mu_{\Omega,q}\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)} \leq C_{p,q} \|f\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)}.$$

Theorem 1.2 [10] Let $1 < q < 2$, $1 < p < \infty$. If $\Omega \in L(\log L^+)^{2/q+\epsilon}(S^{m-1} \times S^{n-1})$ for any $\epsilon > 0$, then

$$\|\mu_{\Omega,q}\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)} \leq C_{p,q} \|f\|_{\dot{F}_p^{0,q}(\mathbb{R}^m \times \mathbb{R}^n)}.$$

Please see [10] for the definition of $\dot{F}_p^{0,q}(\mathbb{R}^m \times \mathbb{R}^n)$. Here the integral operator of Marcinkiewicz type $\mu_{\Omega,q}$ is defined by

$$\mu_{\Omega,q}f(x,y) = \left(\int_{\mathbb{R}^2} |\phi_{s,t} * f(x,y)|^q ds dt \right)^{1/q};$$

and the convolution operator $\phi_{s,t}$ on $\mathbb{R}^m \times \mathbb{R}^n$ is given by

$$\phi_{s,t} * f(x,y) = \frac{1}{2^{s+t}} \int \int_{|u| \leq 2^s, |v| \leq 2^t} \frac{\Omega(u',v')}{|u|^{m-1}|v|^{n-1}} f(x-u, y-v) dudv.$$

The purpose of this paper is to investigate whether or not we can weaken the hypothesis of the kernel Ω for the singular integral defined in the abstract. Although it is known that the kernel $\Omega \in L(\log L^+)^2(S^{n_1-1} \times S^{n_2-1})$ is optimal even when $h \equiv 1$, it turns out that the condition $\Omega \in L(\log L^+)^2(S^{n_1-1} \times S^{n_2-1})$ can be relaxed to $\Omega \in L(\log L^+)^{\nu_q}(S^{n_1-1} \times S^{n_2-1})$ ($1 \leq \nu_q \leq 2$), provided that we impose a stronger condition on the function h . Indeed, if $h \in L^{q'}(\mathbb{R}^+ \times \mathbb{R}^+; \frac{dr_1}{r_1} \frac{dr_2}{r_2})$ ($\frac{1}{q} + \frac{1}{q'} = 1$), where

$$\begin{aligned} & L^{q'}\left(\mathbb{R}^+ \times \mathbb{R}^+; \frac{dr_1}{r_1} \frac{dr_2}{r_2}\right) \\ &= \left\{ h : \|h\|_{L^{q'}(\mathbb{R}^+ \times \mathbb{R}^+; \frac{dr_1}{r_1} \frac{dr_2}{r_2})} := \left(\int_0^\infty \int_0^\infty |h(r_1, r_2)|^{q'} \frac{dr_1}{r_1} \frac{dr_2}{r_2} \right)^{1/q'} < \infty \right\}, \end{aligned}$$

then $\nu_q = 1$ if $q > 2$, and $\nu_q = 2/q$ if $1 < q \leq 2$. We state some definitions and relevant background in Section 2 of this paper. The main result and its proof will be given in Section 3.

2 Preliminary Background

2.1 The Homogeneous Triebel-Lizorkin Spaces $\dot{F}_p^{s,q}(\mathbb{R}^n)$

Fix a radial function $\phi \in C^\infty(\mathbb{R}^n)$ such that $\text{supp}(\phi) \subset \{x \in \mathbb{R}^n : 1/2 < |x| < 2\}$, $0 \leq \phi(x) \leq 1$, $\phi(x) > c > 0$ if $3/5 \leq |x| \leq 5/3$, and $\sum_{j=-\infty}^\infty \phi_j(|x|) = 1$ for all $x \neq 0$, where $\phi_j(x) = \phi(2^j x)$. Define the functions Ψ_j by $\hat{\Psi}_j(\zeta) = \phi_j(\zeta)$, so that $(\widehat{\Psi_j * f})(\zeta) = \phi_j(\zeta)\hat{f}(\zeta)$. For $1 < p, q < \infty$, $s \in \mathbb{R}$, the Triebel-Lizorkin space $\dot{F}_p^{s,q}(\mathbb{R}^n)$ is the space of all tempered

distributions f (modulo polynomials) with the norm defined by

$$\|f\|_{\dot{F}_p^{s,q}(\mathbb{R}^n)} = \left\| \left(\sum_{j=-\infty}^{\infty} |2^{-js} \Psi_j * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

See [14] for more information on this subject.

2.2 The Homogeneous Triebel-Lizorkin Spaces $S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

In view of the definition of the inhomogeneous Triebel-Lizorkin space $S_{p,q}^{\alpha_1, \alpha_2} F(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ defined in [12, 13], we define the space $S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as follows. We split $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$; and for $x, y \in \mathbb{R}^{n_1+n_2}$, we write $x = (x_1, x_2), y = (y_1, y_2), x + y = (x_1 + x_2, y_1 + y_2)$, where $x_i, y_i \in \mathbb{R}^{n_i}, i = 1, 2$. For $i = 1, 2$, fix a Schwartz radial function $\phi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ such that $\text{supp} \hat{\phi}^{(i)} \subset \{\zeta_i \in \mathbb{R}^{n_i} : 1/2 \leq |\zeta_i| \leq 2\}, \hat{\phi}^{(i)}(\zeta_i) \geq 0, \hat{\phi}^{(i)}(\zeta_i) \geq c > 0$ if $3/5 \leq |\zeta_i| \leq 5/3$. For $\tau \in \mathbb{R}$, define $\hat{\phi}_\tau^{(i)}(\zeta_i) = \hat{\phi}^{(i)}(\tau \zeta_i)$. Then $\phi_\tau^{(i)}(x_i) = \tau^{-n_i} \phi^{(i)}(\frac{x_i}{\tau})$, where $x_i \in \mathbb{R}^{n_i}$. Now let $\mathbb{Z}(\mathbb{R}^{n_1+n_2}) = \{f : f \in \mathcal{S}(\mathbb{R}^{n_1+n_2}) \text{ such that } D^\alpha \hat{f}(0) = 0 \text{ for every multi-index } \alpha\}$, and let $\mathbb{Z}'(\mathbb{R}^{n_1+n_2})$ be the topological dual of $\mathbb{Z}(\mathbb{R}^{n_1+n_2})$. For $1 < p, q < \infty, \alpha_1, \alpha_2 \in \mathbb{R}$, we define

$$S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \left\{ f \in \mathbb{Z}'(\mathbb{R}^{n_1+n_2}) : \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} < \infty \right\},$$

where

$$\|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} := \left\| \left(\int_0^\infty \int_0^\infty |t_1^{-\alpha_1} t_2^{-\alpha_2} (\phi_{t_1}^{(1)} \otimes \phi_{t_2}^{(2)}) * f(\cdot, \cdot)|^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n_1+n_2})}.$$

Note that the space $S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is a Banach space for $1 < p, q < \infty$, and is independent of the choice $\phi^{(i)} (i = 1, 2)$ in the sense of equivalent norms. Moreover,

$$\mathbb{Z}(\mathbb{R}^{n_1+n_2}) \subset S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subset \mathbb{Z}'(\mathbb{R}^{n_1+n_2}),$$

and $\mathbb{Z}(\mathbb{R}^{n_1+n_2})$ is dense in $S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Let p' (resp. q') be the conjugate of p (resp. q). Then $(S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))^* = S_{p',q'}^{-\alpha_1, -\alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, where $(S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))^*$ is the dual space of $S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. The proof of the above properties of the space $S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is essentially similar to the proof of the properties of the space $\dot{F}_p^{s,q}(\mathbb{R}^n)$, which can be found in [14].

3 Main Theorem

Let \mathbb{R}^+ denote the interval $(0, \infty)$. For $1 < p < \infty$, let p' stand for the conjugate of p , i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. For $1 \leq q < \infty$, let $\mathcal{C}_q(\mathbb{R}^+ \times \mathbb{R}^+)$ denote the class of all measurable functions b such that

$$\|b\|_{\mathcal{C}_q(\mathbb{R}^+ \times \mathbb{R}^+)} := \sup_{R_1, R_2 > 0} \left\{ \left(\frac{1}{R_1 R_2} \int_0^{R_2} \int_0^{R_1} |b(r_1, r_2)|^q dr_1 dr_2 \right)^{1/q} \right\} < \infty.$$

Note that $L^\infty(\mathbb{R}^+ \times \mathbb{R}^+) \subset \mathcal{C}_q(\mathbb{R}^+ \times \mathbb{R}^+), 1 \leq q < \infty$, and $\mathcal{C}_{q_2}(\mathbb{R}^+ \times \mathbb{R}^+) \subseteq \mathcal{C}_{q_1}(\mathbb{R}^+ \times \mathbb{R}^+)$ for $1 \leq q_1 \leq q_2 \leq \infty$. Also let $\mathcal{H}_q(\mathbb{R}^+ \times \mathbb{R}^+)$ stand for the class of all measurable functions b such that

$$\|b\|_{\mathcal{H}_q(\mathbb{R}^+ \times \mathbb{R}^+)} := \left(\int_0^\infty \int_0^\infty |b(r_1, r_2)|^q \frac{dr_1}{r_1} \frac{dr_2}{r_2} \right)^{1/q} < \infty.$$

For notational convenience, we use the notation $\mathcal{H}_q(\mathbb{R}^+ \times \mathbb{R}^+)$ rather than $L^q(\mathbb{R}^+ \times \mathbb{R}^+; \frac{dr_1}{r_1} \frac{dr_2}{r_2})$.

In the sequel, we split $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. For $x, y \in \mathbb{R}^{n_1+n_2}$, we write $x = (x_1, x_2)$, $y = (y_1, y_2)$, and of course $x + y = (x_1 + x_2, y_1 + y_2)$, where $x_i, y_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. Also the letter C denotes a positive constant, which may vary at different occurrences. However, it does not depend on any essential variable.

Theorem 1 Let h be a measurable function defined on $\mathbb{R}^+ \times \mathbb{R}^+$. Let Ω be an integrable function on $S^{n_1-1} \times S^{n_2-1}$ ($n_1, n_2 \geq 2$), which is homogeneous of degree zero with respect to both variables and satisfies

- a) $\int_{S^{n_1-1}} \Omega(u, v) d\mu_1(u) = 0 \forall v \in S^{n_2-1}$, and
- b) $\int_{S^{n_2-1}} \Omega(u, v) d\mu_2(v) = 0 \forall u \in S^{n_1-1}$.

For $f \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$, define the singular integral Tf by

$$Tf(x_1, x_2) = \text{p. v. } K * f(x_1, x_2) \\ = \text{p. v. } \int \int_{\mathbb{R}^{n_1+n_2}} \frac{\Omega(y'_1, y'_2) h(|y_1|, |y_2|)}{|y_1|^{n_1} |y_2|^{n_2}} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2.$$

Set

$$\nu_q = \begin{cases} 2 & \text{if } h \in \mathcal{C}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+), \\ 1 & \text{if } h \in \mathcal{H}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+), \text{ and } q > 2, \\ 2/q & \text{if } h \in \mathcal{H}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+), \text{ and } 1 < q \leq 2. \end{cases}$$

If $\Omega \in L(\log L^+)^{\nu_q}(S^{n_1-1} \times S^{n_2-1})$, then for any $p, q \in (1, \infty)$, $\alpha_1, \alpha_2 \in \mathbb{R}$, T has a bounded extension on $S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Precisely,

$$\|Tf\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C_{p,q} \|\Omega\|_{L(\log L^+)^{\nu_q}(S^{n_1-1} \times S^{n_2-1})} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$$

for $1 < p, q < \infty$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Proof The proof of this theorem follows some basic ideas in [6]. In order to prove the theorem, we need the following lemmas. □

Lemma 1 Let μ be the product measures on $S^{n_1-1} \times S^{n_2-1}$ ($n_1, n_2 \geq 2$). Let μ_1 and μ_2 be the measures on S^{n_1-1} and S^{n_2-1} respectively, which are induced from the product measure μ . We normalize μ so that $\mu(S^{n_1-1} \times S^{n_2-1}) = \mu_1(S^{n_1-1})\mu_2(S^{n_2-1}) = 1$. Let $E_o = \{(u, v) \in S^{n_1-1} \times S^{n_2-1} : |\Omega(u, v)| \leq 2\}$, and for $l \in \mathbb{N}$, let $E_l = \{(u, v) \in S^{n_1-1} \times S^{n_2-1} : 2^l < |\Omega(u, v)| \leq 2^{l+1}\}$. Let $A(\Omega) = \{l \in \mathbb{N} : \mu(E_l) > 2^{-4l}\}$. For $l \in A(\Omega)$, define

$$\Omega_l^{(1)}(u) = \int_{S^{n_2-1}} \Omega(u, v) \chi_{E_l}(u, v) d\mu_2(v)$$

and

$$\Omega_l^{(2)}(v) = \int_{S^{n_1-1}} \Omega(u, v) \chi_{E_l}(u, v) d\mu_1(u).$$

Now define a_l ($l \in A(\Omega)$) on $S^{n_1-1} \times S^{n_2-1}$ by

$$a_l(u, v) = \frac{1}{\|\Omega\|_{L^1(E_l)}} \left\{ \Omega(u, v) \chi_{E_l}(u, v) - \mu_1(S^{n_1-1}) \Omega_l^{(1)}(u) - \mu_2(S^{n_2-1}) \Omega_l^{(2)}(v) \right\} \\ + \frac{1}{\|\Omega\|_{L^1(E_l)}} \int \int_{S^{n_1-1} \times S^{n_2-1}} \Omega(u, v) \chi_{E_l}(u, v) d\mu_1(u) d\mu_2(v).$$

Then each a_l satisfies the cancellation conditions as Ω does. Moreover, $\|a_l\|_{L^1(S^{n_1-1} \times S^{n_2-1})} \leq 4$ and $\|a_l\|_{L^2(S^{n_1-1} \times S^{n_2-1})} \leq 2^{2l+3}$. Now let

$$\Omega_o = \Omega - \sum_{l \in A(\Omega)} \|\Omega\|_{L^1(E_l)} a_l \equiv \Omega - \sum_{l \in A(\Omega)} \Omega_l.$$

Then Ω_o also satisfies the cancellation conditions and $\|\Omega_o\|_{L^2(S^{n_1-1} \times S^{n_2-1})} \leq C$. Consequently, Ω has a decomposition

$$\Omega = \Omega_o + \sum_{l \in A(\Omega)} \Omega_l.$$

The proof of Lemma 1 is straightforward, and can be found in [1, 7]. The decomposition of Ω leads to a decomposition of the corresponding operators:

$$T = T_o + \sum_{l \in A(\Omega)} T_l, \tag{3.1}$$

where

$$T_l f(x) = \text{p.v.} \int \int_{\mathbb{R}^{n_1+n_2}} \frac{h(|y_1|, |y_2|) \Omega_l(y'_1, y'_2)}{|y_1|^{n_1} |y_2|^{n_2}} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2,$$

and similar definition for $T_o f(x)$ (with Ω_o in place of Ω_l). We will show that

$$\|T_o f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C_{p,q} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}, \tag{3.2}$$

$$\|T_l f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C_{p,q} l^{\nu_q} \|\Omega\|_{L^1(E_l)} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}, \tag{3.3}$$

for $1 < p, q < \infty$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. An application of Minkowski's inequality will yield that

$$\begin{aligned} \|Tf\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} &\leq \|T_o f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} + \sum_{l \in A(\Omega)} \|T_l f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\ &\leq C_{p,q} \left(1 + \sum_{l \in A(\Omega)} l^{\nu_q} \|\Omega\|_{L^1(E_l)} \right) \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\ &\leq C_{p,q} \left(1 + \|\Omega\|_{L(\log L)^{\nu_q}(S^{n_1-1} \times S^{n_2-1})} \right) \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}. \end{aligned}$$

The proof of inequality (3.2) is essentially the same as the proof of inequality (3.3). Just set $l = 1$ and replace Ω_l by Ω_o in the proof of (3.3), we will obtain (3.2). Therefore to prove the theorem, it is enough to prove inequality (3.3). For $i = 1, 2$, let $\hat{\phi}^{(i)}$ be a radial function in $\mathcal{S}(\mathbb{R}^{n_i})$ such that $\text{supp} \hat{\phi}^{(i)} \subset \{\zeta_i \in \mathbb{R}^{n_i} : 1/2 \leq |\zeta_i| \leq 2\}$, $0 \leq \hat{\phi}^{(i)}(\zeta_i) \leq 1$, $\hat{\phi}^{(i)}(\zeta_i) \geq c > 0$ if $3/5 \leq |\zeta_i| \leq 5/3$, and $\int_{\mathbb{R}^{n_i}} \left\{ \hat{\phi}^{(i)}(2^t \zeta_i) \right\}^2 dt = 1$ for all $\zeta_i \neq 0 \in \mathbb{R}^{n_i}$. For $\tau \in \mathbb{R}$, define $\hat{\phi}_\tau^{(i)}(\zeta_i) = \hat{\phi}^{(i)}(\tau \zeta_i)$. Now set $\Psi = \phi^{(1)} \otimes \phi^{(2)}$ and let $\Psi_{2^t, 2^s}(x_1, x_2) = \phi_{2^t}^{(1)}(x_1) \otimes \phi_{2^s}^{(2)}(x_2) = 2^{-n_1 t} 2^{-n_2 s} \phi^{(1)}(2^{-t} x_1) \otimes \phi^{(2)}(2^{-s} x_2)$, where $x_i \in \mathbb{R}^{n_i}$ ($i = 1, 2$). Note that $\hat{\Psi}_{2^t, 2^s}(\zeta_1, \zeta_2) = \hat{\phi}_{2^t}^{(1)}(\zeta_1) \hat{\phi}_{2^s}^{(2)}(\zeta_2)$, and $\int \int_{\mathbb{R} \times \mathbb{R}} \hat{\Psi}_{2^t, 2^s}^2(\zeta_1, \zeta_2) dt ds = 1$ for all $(\zeta_1, \zeta_2) \neq 0 \in \mathbb{R}^{n_1+n_2}$. For $f \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$, denote $S_{2^t, 2^s} f = \Psi_{2^t, 2^s} * f$, and observe that for any fixed $l \in \mathbb{N}$, we have $f = l^2 \int \int_{\mathbb{R} \times \mathbb{R}} S_{2^{lt}, 2^{ls}} S_{2^{lt}, 2^{ls}} f dt ds$. For $\tau \in \mathbb{R}$, denote $\chi_{2^{l\tau}}^{(i)}(y_i)$ to be the characteristic function on the set $\{y_i \in \mathbb{R}^{n_i} : 2^{l\tau} \leq |y_i| < 2^{l(\tau+1)}\}$. Observe that

$$\begin{aligned} &T_l f(x_1, x_2) \\ &= \int \int_{\mathbb{R}^2} \int \int_{\mathbb{R}^{n_1+n_2}} \frac{\Omega_l(y'_1, y'_2) h(|y_1|, |y_2|)}{|y_1|^{n_1} |y_2|^{n_2}} f(x_1 - y_1, x_2 - y_2) \chi_{2^{lt}}^{(1)}(y_1) \chi_{2^{ls}}^{(2)}(y_2) dy_1 dy_2 dt ds \end{aligned}$$

$$\equiv \int_{\mathbb{R}^2} \sigma_{2^{lt}, 2^{ls}} * f(x_1, x_2) dt ds, \tag{3.4}$$

where

$$\begin{aligned} & \hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2) \\ &= \int_{\mathbb{R}^{n_1+n_2}} \frac{\Omega_l(y'_1, y'_2) h(|y_1|, |y_2|)}{|y_1|^{n_1} |y_2|^{n_2}} e^{i(\xi_1 \cdot y_1 + \xi_2 \cdot y_2)} \chi_{2^{lt}}^{(1)}(y_1) \chi_{2^{ls}}^{(2)}(y_2) dy_1 dy_2 \\ &= \int_{2^{2^{ls}}}^{2^{2^{(s+1)}}} \int_{2^{2^{lt}}}^{2^{2^{(t+1)}}} \int_{S^{n_1-1} \times S^{n_2-1}} \Omega_l(u, v) h(r_1, r_2) e^{i(\xi_1 \cdot r_1 u + \xi_2 \cdot r_2 v)} du dv \frac{dr_1}{r_1} \frac{dr_2}{r_2}. \end{aligned} \tag{3.5}$$

For notational convenience, we denote $b_l = \|\Omega\|_{L^1(E_l)}$, $\delta_q = \max\{q, 2\}$, and $\|h\|_{q'} = \left(\int_{2^{2^{ls}}}^{2^{2^{(s+1)}}} \int_{2^{2^{lt}}}^{2^{2^{(t+1)}}} |h(r_1, r_2)|^{q'} \frac{dr_1}{r_1} \frac{dr_2}{r_2}\right)^{1/q'}$. Observe that if $h \in \mathcal{H}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+)$, then $\|h\|_{q'} \leq \|h\|_{\mathcal{H}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+)} \leq C < \infty$. If $h \in \mathcal{C}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+)$, then $\|h\|_{q'} \leq 4l^{2/q'} \|h\|_{\mathcal{C}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+)} \leq Cl^{2/q'}$. In either case, the bound of $\|h\|_{q'}$ does not depend on $t, s \in \mathbb{R}$. We have the following lemma.

Lemma 2

$$|\hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2)| \leq C_q b_l l^{2/q} \|h\|_{q'} |2^{l(t+1)} \xi_1| |2^{l(s+1)} \xi_2| \tag{3.6}$$

$$\leq C_q b_l l^{2/q} \|h\|_{q'} |2^{l(t+1)} \xi_1| |2^{ls} \xi_2|^{-\frac{1}{4\delta_q t}} \tag{3.7}$$

$$\leq C_q b_l l^{2/q} \|h\|_{q'} |2^{lt} \xi_1|^{-\frac{1}{4\delta_q t}} |2^{l(s+1)} \xi_2| \tag{3.8}$$

$$\leq C_q b_l l^{2/q} \|h\|_{q'} |2^{lt} \xi_1|^{-\frac{1}{4\delta_q t}} |2^{ls} \xi_2|^{-\frac{1}{4\delta_q t}}. \tag{3.9}$$

Proof Following the notations in [8], we denote

$$\Delta_{\xi_1}^1 \hat{\sigma}_{2^{lt}, 2^{ls}}(0, \xi_2) = \hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2) - \hat{\sigma}_{2^{lt}, 2^{ls}}(0, \xi_2),$$

$$\Delta_{\xi_2}^2 \hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, 0) = \hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2) - \hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, 0),$$

$$\Delta_{\xi_1, \xi_2}^{1,2} \hat{\sigma}_{2^{lt}, 2^{ls}}(0, 0) = \{\hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2) - \hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, 0)\} - \{\hat{\sigma}_{2^{lt}, 2^{ls}}(0, \xi_2) - \hat{\sigma}_{2^{lt}, 2^{ls}}(0, 0)\}.$$

Recall that $\Omega_l = \|\Omega\|_{L^1(E_l)} a_l = b_l a_l$. By the cancellation conditions on Ω_l , we have

$$\begin{aligned} |\hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2)| &= |\Delta_{\xi_1, \xi_2}^{1,2} \hat{\sigma}_{2^{lt}, 2^{ls}}(0, 0)| \\ &\leq \int_{2^{2^{ls}}}^{2^{2^{(s+1)}}} \int_{2^{2^{lt}}}^{2^{2^{(t+1)}}} \int_{S^{n_1-1} \times S^{n_2-1}} |\Omega_l(u, v) h(r_1, r_2)| \\ &\quad \times |(e^{i\xi_1 \cdot r_1 u} - 1)(e^{i\xi_2 \cdot r_2 v} - 1)| du dv \frac{dr_1}{r_1} \frac{dr_2}{r_2} \\ &\leq C b_l l^{2/q} \|h\|_{q'} |2^{l(t+1)} \xi_1| |2^{l(s+1)} \xi_2|. \end{aligned} \tag{3.10}$$

Also,

$$\begin{aligned} |\hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2)| &= |\Delta_{\xi_1}^1 \hat{\sigma}_{2^{lt}, 2^{ls}}(0, \xi_2)| \\ &\leq b_l \int_{2^{2^{lt}}}^{2^{2^{(t+1)}}} \int_{S^{n_1-1}} \left| \int_{2^{2^{ls}}}^{2^{2^{(s+1)}}} \int_{S^{n_2-1}} a_l(u, v) h(r_1, r_2) e^{i\xi_2 \cdot r_2 v} dv \frac{dr_2}{r_2} \right| \\ &\quad \times |e^{i\xi_1 \cdot r_1 u} - 1| du \frac{dr_1}{r_1}. \end{aligned} \tag{3.11}$$

Denote $I(r_1, u) = \int_{2^{ls}}^{2^{l(s+1)}} \int_{S^{n_2-1}} a_l(u, v)h(r_1, r_2)e^{i\xi_2 \cdot r_2 v} dv \frac{dr_2}{r_2}$, $a_l^{(1)}(u) = \int_{S^{n_2-1}} |a_l(u, v)|dv$. Then $\int_{S^{n_1-1}} a_l^{(1)}(u)du = \|a_l\|_{L^1(S^{n_1-1} \times S^{n_2-1})}$. We may assume without loss of generality that $a_l^{(1)}(u) \neq 0 \forall u \in S^{n_1-1}$. For if $B = \{u \in S^{n_1-1} : a_l^{(1)}(u) = 0\}$, then we can split the integral $\int_{S^{n_1-1}} \dots$ on the right hand side of inequality (3.11) into the sum of two integrals $\int_{S^{n_1-1} \setminus B} \dots$ and $\int_B \dots$. It follows by a direct integration that the value of the integral on the right hand side of (3.11), which involves the integral $\int_B \dots$, is zero. Now denote $w_l(u, v) = \frac{a_l(u, v)}{a_l^{(1)}(u)}$. Then

$$\begin{aligned} & |I(r_1, u)| \\ &= a_l^{(1)}(u) \left| \int_{2^{ls}}^{2^{l(s+1)}} \int_{S^{n_2-1}} w_l(u, v)h(r_1, r_2)e^{i\xi_2 \cdot r_2 v} dv \frac{dr_2}{r_2} \right| \\ &\leq a_l^{(1)}(u) \left(\int_{2^{ls}}^{2^{l(s+1)}} |h(r_1, r_2)|^{q'} \frac{dr_2}{r_2} \right)^{1/q'} \left(\int_{2^{ls}}^{2^{l(s+1)}} \left| \int_{S^{n_2-1}} w_l(u, v)e^{i\xi_2 \cdot r_2 v} dv \right|^q \frac{dr_2}{r_2} \right)^{1/q} \\ &\equiv a_l^{(1)}(u) \left(\int_{2^{ls}}^{2^{l(s+1)}} |h(r_1, r_2)|^{q'} \frac{dr_2}{r_2} \right)^{1/q'} J(u, \xi_2). \end{aligned}$$

If $q \geq 2$, then since $\left| \int_{S^{n_2-1}} w_l(u, v)e^{i\xi_2 \cdot r_2 v} dv \right| \leq 1$, we have

$$\begin{aligned} J^q(u, \xi_2) &\leq \int_{2^{ls}}^{2^{l(s+1)}} \left| \int_{S^{n_2-1}} w_l(u, v)e^{i\xi_2 \cdot r_2 v} dv \right|^2 \frac{dr_2}{r_2} \\ &= \int_{S^{n_2-1}} \int_{S^{n_2-1}} w_l(\cdot, v)\bar{w}_l(\cdot, \tilde{v}) \left| \int_{2^{ls}}^{2^{l(s+1)}} e^{i\xi_2 \cdot r_2(v-\tilde{v})} \frac{dr_2}{r_2} \right| dv d\tilde{v} \\ &\leq Cl|2^{ls}\xi_2|^{-1/4} \int_{S^{n_2-1}} \int_{S^{n_2-1}} w_l(\cdot, v)\bar{w}_l(\cdot, \tilde{v})|\xi_2' \cdot (v-\tilde{v})|^{-1/4} dv d\tilde{v} \\ &\leq Cl|2^{ls}\xi_2|^{-1/4} \int_{S^{n_2-1}} |w_l(u, v)|^2 dv, \end{aligned} \tag{3.12}$$

where the second inequality follows from the fact that

$$\begin{aligned} \left| \int_{2^{ls}}^{2^{l(s+1)}} e^{i\xi_2 \cdot r_2(v-\tilde{v})} \frac{dr_2}{r_2} \right| &\leq Cl \min \{1, |2^{ls}\xi_2||\xi_2' \cdot (v-\tilde{v})|^{-1}\} \\ &\leq Cl|2^{ls}\xi_2|^{-1/4}|\xi_2' \cdot (v-\tilde{v})|^{-1/4}. \end{aligned}$$

So $J(u, \xi_2) \leq Cl^{1/q}|2^{ls}\xi_2|^{-\frac{1}{4q}} \left(\int_{S^{n_2-1}} |w_l(u, v)|^2 dv \right)^{1/q}$. A direct integration yields $J(u, \xi_2) \leq Cl^{1/q}$. Thus

$$\begin{aligned} J(u, \xi_2) &\leq Cl^{1/q} \min \left\{ 1, |2^{ls}\xi_2|^{-\frac{1}{4q}} \left(\int_{S^{n_2-1}} |w_l(u, v)|^2 dv \right)^{1/q} \right\} \\ &\leq Cl^{1/q}|2^{ls}\xi_2|^{-\frac{1}{4q}} \left(\int_{S^{n_2-1}} |w_l(u, v)|^2 dv \right)^{\frac{1}{q}}. \end{aligned}$$

Therefore,

$$I(r_1, u) \leq Cl^{1/q}|2^{ls}\xi_2|^{-\frac{1}{4q}} a_l^{(1)}(u) \left(\int_{S^{n_2-1}} |w_l(u, v)|^2 dv \right)^{\frac{1}{q}} \left(\int_{2^{ls}}^{2^{l(s+1)}} |h(r_1, r_2)|^{q'} \frac{dr_2}{r_2} \right)^{1/q'}$$

and

$$|\hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2)| \leq C b_l l^{1/q} |2^{l(t+1)} \xi_1| |2^{ls} \xi_2|^{-\frac{1}{4qt}} \left\{ \int_{2^{lt}}^{2^{l(t+1)}} \left(\int_{2^{ls}}^{2^{l(s+1)}} |h(r_1, r_2)|^{q'} \frac{dr_2}{r_2} \right)^{1/q'} \frac{dr_1}{r_1} \right\} \\ \times \left\{ \int_{S^{n_1-1}} a_l^{(1)}(u) \left(\int_{S^{n_2-1}} |w_l(u, v)|^2 dv \right)^{\frac{1}{qt}} du \right\}.$$

By Hölder's inequality, we have

$$\int_{2^{lt}}^{2^{l(t+1)}} \left(\int_{2^{ls}}^{2^{l(s+1)}} |h(r_1, r_2)|^{q'} \frac{dr_2}{r_2} \right)^{1/q'} \frac{dr_1}{r_1} \leq C l^{1/q} \|h\|_{q'}, \\ \int_{S^{n_1-1}} a_l^{(1)}(u) \left(\int_{S^{n_2-1}} |w_l(u, v)|^2 dv \right)^{\frac{1}{qt}} du \\ = \int_{S^{n_1-1}} |a_l^{(1)}(u)|^{1-\frac{2}{qt}} \left(\int_{S^{n_2-1}} |a_l(u, v)|^2 dv \right)^{\frac{1}{qt}} du \\ \leq \|a_l\|_{L^2(S^{n_1-1} \times S^{n_2-1})}^{\frac{2}{qt}} \left(\int_{S^{n_1-1}} |a_l^{(1)}(u)|^{\frac{qt-2}{qt-1}} du \right)^{\frac{qt-1}{qt}} \\ \leq \|a_l\|_{L^2(S^{n_1-1} \times S^{n_2-1})}^{\frac{2}{qt}} \|a_l\|_{L^1(S^{n_1-1} \times S^{n_2-1})}^{\frac{qt-2}{qt}} \{ \mu_1(S^{n_1-1}) \}^{\frac{1}{qt-1}} \leq C_q.$$

Therefore,

$$|\hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2)| \leq C_q b_l l^{2/q} \|h\|_{q'} |2^{l(t+1)} \xi_1| |2^{ls} \xi_2|^{-\frac{1}{4qt}} \text{ if } q \geq 2.$$

If $1 < q < 2$, then it follows from Hölder's inequality that

$$J(u, \xi_2) \leq C l^{(\frac{1}{q}-\frac{1}{2})} \left(\int_{2^{ls}}^{2^{l(s+1)}} \left| \int_{S^{n_2-1}} w_l(u, v) e^{i\xi_2 \cdot r_2 v} dv \right|^2 \frac{dr_2}{r_2} \right)^{1/2} \\ \leq C l^{1/q} \min \left\{ 1, |2^{ls} \xi_2|^{-\frac{1}{8}} \left(\int_{S^{n_2-1}} |w_l(u, v)|^2 dv \right)^{1/2} \right\} \\ \leq C l^{1/q} |2^{ls} \xi_2|^{-\frac{1}{8t}} \left(\int_{S^{n_2-1}} |w_l(u, v)|^2 dv \right)^{\frac{1}{2t}}.$$

By using similar arguments as above, we obtain

$$|\hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2)| \leq C_q b_l l^{2/q} \|h\|_{q'} |2^{l(t+1)} \xi_1| |2^{ls} \xi_2|^{-\frac{1}{8t}} \text{ if } 1 < q < 2.$$

Thus inequality (3.7) of Lemma 2 is proved. By symmetry, we also obtain inequality (3.8). It remains to prove inequality (3.9). Denote $c_l(u, v) = \|a_l\|_{L^1(S^{n_1-1} \times S^{n_2-1})}^{-1} a_l(u, v)$, and note that $\|c_l\|_{L^1(S^{n_1-1} \times S^{n_2-1})} = 1$. An application of Hölder's inequality produces

$$|\hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2)| \\ = b_l \|a_l\|_{L^1} \left| \int_{2^{ls}}^{2^{l(s+1)}} \int_{2^{lt}}^{2^{l(t+1)}} h(r_1, r_2) \int \int_{S^{n_1-1} \times S^{n_2-1}} \Omega_l(u, v) c_l(u, v) e^{i(\xi_1 \cdot r_1 u + \xi_2 \cdot r_2 v)} dudv \frac{dr_1}{r_1} \frac{dr_2}{r_2} \right| \\ \leq C b_l \|h\|_{q'} \left(\int_{2^{ls}}^{2^{l(s+1)}} \int_{2^{lt}}^{2^{l(t+1)}} \left| \int \int_{S^{n_1-1} \times S^{n_2-1}} \Omega_l(u, v) c_l(u, v) e^{i(\xi_1 \cdot r_1 u + \xi_2 \cdot r_2 v)} dudv \left| \frac{dr_1}{r_1} \frac{dr_2}{r_2} \right|^q \right)^{1/q} \right)^{1/q}.$$

Again, by using similar arguments as in the previous case, we finally obtain

$$\begin{aligned} |\hat{\sigma}_{2^{lt}, 2^{ls}}(\xi_1, \xi_2)| &\leq C_q b_l l^{2/q} \|h\|_{q'} |2^{lt}\xi_1|^{-\frac{1}{4qt}} |2^{ls}\xi_2|^{-\frac{1}{4qt}} \text{ if } q \geq 2, \\ &\leq C_q b_l l^{2/q} \|h\|_{q'} |2^{lt}\xi_1|^{-\frac{1}{8t}} |2^{ls}\xi_2|^{-\frac{1}{8t}} \text{ if } 1 < q < 2. \end{aligned}$$

Lemma 2 is proved. □

Now observe that

$$\begin{aligned} T_l f(x_1, x_2) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\sigma_{2^{lt}, 2^{ls}} * f)(x_1, x_2) dt ds \\ &= l^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sigma_{2^{lt}, 2^{ls}} * \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} S_{2^{l(t+u)}, 2^{l(s+v)}} S_{2^{l(t+u)}, 2^{l(s+v)}} f(x_1, x_2) dudv \right) dt ds \\ &= l^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} S_{2^{l(t+u)}, 2^{l(s+v)}} (\sigma_{2^{lt}, 2^{ls}} * S_{2^{l(t+u)}, 2^{l(s+v)}} f(x_1, x_2)) dt ds \right) dudv \\ &\equiv \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} T_l^{u,v} f(x_1, x_2) dudv, \end{aligned} \tag{3.13}$$

where

$$T_l^{u,v} f = l^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} S_{2^{l(t+u)}, 2^{l(s+v)}} (\sigma_{2^{lt}, 2^{ls}} * S_{2^{l(t+u)}, 2^{l(s+v)}} f) dt ds. \tag{3.14}$$

Note that

$$\begin{aligned} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} &\cong \left\| \left(\int_0^\infty \int_0^\infty |t^{-\alpha_1} s^{-\alpha_2} \Psi_{t,s} * f|^q \frac{dt ds}{t s} \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n_1+n_2})} \\ &\cong l^{2/q} \left\| \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |2^{-\alpha_1 l t} 2^{-\alpha_2 l s} \Psi_{2^{lt}, 2^{ls}} * f|^q dt ds \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n_1+n_2})}. \end{aligned}$$

For any $g \in S_{p',q'}^{-\alpha_1, -\alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, we have

$$\begin{aligned} &|\langle T_l^{u,v} f, g \rangle| \\ &= l^2 \left| \int_{\mathbb{R}^{n_1+n_2}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} S_{2^{l(t+u)}, 2^{l(s+v)}} (\sigma_{2^{lt}, 2^{ls}} * S_{2^{l(t+u)}, 2^{l(s+v)}} f(x_1, x_2)) g(x_1, x_2) dt ds dx_1 dx_2 \right| \\ &\leq l^2 \int_{\mathbb{R}^{n_1+n_2}} \left| \int_{\mathbb{R}^2} \sigma_{2^{lt}, 2^{ls}} * S_{2^{l(t+u)}, 2^{l(s+v)}} f(x_1, x_2) \tilde{S}_{2^{l(t+u)}, 2^{l(s+v)}} g(x_1, x_2) dt ds \right| dx_1 dx_2 \\ &\leq l^2 \left\| \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} \sigma_{2^{lt}, 2^{ls}} * S_{2^{l(t+u)}, 2^{l(s+v)}} f|^q dt ds \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n_1+n_2})} \\ &\quad \times \left\| \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |2^{\alpha_1 l(t+u)} 2^{\alpha_2 l(s+v)} \tilde{S}_{2^{l(t+u)}, 2^{l(s+v)}} g|^{q'} dt ds \right)^{1/q'} \right\|_{L^{p'}(\mathbb{R}^{n_1+n_2})} \end{aligned}$$

$$\leq l^{2/q} \left\| \left(\int_{\mathbb{R}^2} \int |2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} \sigma_{2^l t, 2^l s} * S_{2^l(t+u), 2^l(s+v)} f|^q dt ds \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n_1+n_2})} \\ \times \|g\|_{S_{p',q'}^{-\alpha_1, -\alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})},$$

where $\tilde{S}_{2^l(t+u), 2^l(s+v)} g(x_1, x_2) = S_{2^l(t+u), 2^l(s+v)} \tilde{g}(-x_1, -x_2)$, and $\tilde{g}(x_1, x_2) = g(-x_1, -x_2)$.

Taking the supremum over all $g \in S_{p',q'}^{-\alpha_1, -\alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with norm ≤ 1 yields

$$\|T_l^{u,v} f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\ \leq C l^{2/q} \left\| \left(\int_{\mathbb{R}^2} \int |2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} \sigma_{2^l t, 2^l s} * S_{2^l(t+u), 2^l(s+v)} f|^q dt ds \right)^{1/q} \right\|_{L^p(\mathbb{R}^{n_1+n_2})} \quad (3.15)$$

When $p = q = 2$, we have

$$\|T_l^{u,v} f\|_{S_{2,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^2 \leq C l^2 \int_{\mathbb{R}^2} \int_{D_{t+u,s+v}} \int |2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} \hat{\sigma}_{2^l t, 2^l s}(\xi_1, \xi_2) \\ \times \hat{\phi}^{(1)}(2^l(t+u)\xi_1) \hat{\phi}^{(2)}(2^l(s+v)\xi_2) \hat{f}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 dt ds,$$

where $D_{t+u,s+v} = \{(\xi_1, \xi_2) \in \mathbb{R}^{n_1+n_2} : 1/2 \leq |2^l(t+u)\xi_1|, |2^l(s+v)\xi_2| \leq 2\}$. If we apply Lemma 2 to the above inequality, the results become

$$\|T_l^{u,v} f\|_{S_{2,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\ \leq \begin{cases} C_q \|h\|_{q'} b_l l^{2/q} 2^{-l(u-1)} 2^{-l(v-1)} \|f\|_{S_{2,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}; & u, v \geq 1 \\ C_q \|h\|_{q'} b_l l^{2/q} 2^{-l(u-1)} 2^{\frac{u}{4\delta q}} \|f\|_{S_{2,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}; & u \geq 1, v < 1 \\ C_q \|h\|_{q'} b_l l^{2/q} 2^{\frac{u}{4\delta q}} 2^{-l(v-1)} \|f\|_{S_{2,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}; & u < 1, v \geq 1 \\ C_q \|h\|_{q'} b_l l^{2/q} 2^{\frac{u}{4\delta q}} 2^{\frac{v}{4\delta q}} \|f\|_{S_{2,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}; & u, v < 1. \end{cases} \quad (3.16)$$

In order to estimate the norm $\|T_l^{u,v} f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$, we need the following lemma.

Lemma 3 Define $L_{t,s}(f)$ by

$$L_{t,s}(f)(x_1, x_2) = \int_{\mathbb{R}^{n_1+n_2}} \frac{|\Omega_l(y'_1, y'_2)|}{|y_1|^{n_1} |y_2|^{n_2}} f(x_1 - y_1, x_2 - y_2) \chi_{2^l t}^{(1)}(y_1) \chi_{2^l s}^{(2)}(y_2) dy_1 dy_2.$$

Then

$$|\sigma_{2^l t, 2^l s} * S_{2^l(t+u), 2^l(s+v)} f(x_1, x_2)| \leq C b_l^{1/q'} \|h\|_{q'} (L_{t,s}(|S_{2^l(t+u), 2^l(s+v)} f|^q)(x_1, x_2))^{1/q}. \quad (3.17)$$

$$\|\sigma_{2^l t, 2^l s} * S_{2^l(t+u), 2^l(s+v)} f\|_{L^q(\mathbb{R}^{n_1+n_2})} \leq C b_l l^{2/q} \|h\|_{q'} \|S_{2^l(t+u), 2^l(s+v)} f\|_{L^q(\mathbb{R}^{n_1+n_2})}. \quad (3.18)$$

$$\| \sup_{t, s \in \mathbb{R}} L_{t,s}(f) \|_{L^p(\mathbb{R}^{n_1+n_2})} \leq C b_l l^2 \|f\|_{L^p(\mathbb{R}^{n_1+n_2})} \text{ for } 1 < p < \infty. \quad (3.19)$$

Proof By Hölder's inequality, we have

$$|\sigma_{2^l t, 2^l s} * S_{2^l(t+u), 2^l(s+v)} f(x_1, x_2)| \\ \leq \left(\int_{\mathbb{R}^{n_1+n_2}} \frac{|h(|y_1|, |y_2|)|^{q'}}{|y_1|^{n_1} |y_2|^{n_2}} |\Omega_l(y'_1, y'_2)| \chi_{2^l t}^{(1)}(y_1) \chi_{2^l s}^{(2)}(y_2) dy_1 dy_2 \right)^{1/q'}$$

$$\begin{aligned} & \times \left(\int \int_{\mathbb{R}^{n_1+n_2}} \frac{|\Omega_l(y'_1, y'_2)|}{|y_1|^{n_1}|y_2|^{n_2}} |S_{2^{l(t+u)}, 2^{l(s+v)}} f(x_1 - y_1, x_2 - y_2)|^q \chi_{2^{lt}}^{(1)}(y_1) \chi_{2^{ls}}^{(2)}(y_2) dy_1 dy_2 \right)^{1/q} \\ & \leq C \|h\|_{q'} b_l^{1/q'} (L_{t,s}(|S_{2^{l(t+u)}, 2^{l(s+v)}} f|^q)(x_1, x_2))^{1/q}, \end{aligned}$$

and a direct integration gives inequality (3.18).

Observe that

$$\begin{aligned} & L_{t,s}(f)(x_1, x_2) \\ &= \int \int_{\mathbb{R}^{n_1+n_2}} \frac{|\Omega_l(y'_1, y'_2)|}{|y_1|^{n_1}|y_2|^{n_2}} |f(x_1 - y_1, x_2 - y_2)| \chi_{2^{lt}}^{(1)}(y_1) \chi_{2^{ls}}^{(2)}(y_2) dy_1 dy_2 \\ &= \int \int_{S^{n_1-1} \times S^{n_2-1}} |\Omega_l(y'_1, y'_2)| \int_{2^{ls}}^{2^{l(s+1)}} \int_{2^{lt}}^{2^{l(t+1)}} |f(x_1 - r_1 y'_1, x_2 - r_2 y'_2)| \frac{dr_1}{r_1} \frac{dr_2}{r_2} d\mu_1(y'_1) d\mu_2(y'_2) \\ &\leq C l^2 \int \int_{S^{n_1-1} \times S^{n_2-1}} |\Omega_l(y'_1, y'_2)| \\ &\quad \times \sup_{\rho_1, \rho_2 > 0} \left\{ \frac{1}{\rho_1 \rho_2} \int_0^{\rho_2} \int_0^{\rho_1} |f(x_1 - r_1 y'_1, x_2 - r_2 y'_2)| dr_1 dr_2 \right\} d\mu_1(y'_1) d\mu_2(y'_2) \\ &\equiv C l^2 \int \int_{S^{n_1-1} \times S^{n_2-1}} |\Omega_l(y'_1, y'_2)| M_2^{y'_2} \circ M_1^{y'_1} f(x_1, x_2) d\mu_1(y'_1) d\mu_2(y'_2), \end{aligned}$$

where $M_i^{y'_i}$ denotes the Hardy-Littlewood maximal function acting on the i -th variable. Observe that $\|M_2^{y'_2} \cdot M_1^{y'_1} f\|_{L^p(\mathbb{R}^{n_1+n_2})} \leq C \|f\|_{L^p(\mathbb{R}^{n_1+n_2})}$ for $1 < p \leq \infty$, and the bound C is independent of the unit vectors $y'_i \in S^{n_i-1}$ ($i = 1, 2$). Therefore by Minkowski's inequality, we obtain inequality (3.19). Lemma 3 is proved.

When $p = q$, it follows from inequality (3.15) that

$$\begin{aligned} & \|T_l^{u,v} f\|_{S_{q,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\ & \leq C l^{2/q} \left\| \left(\int \int_{\mathbb{R}^2} |2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} \sigma_{2^{lt}, 2^{ls}} * S_{2^{l(t+u)}, 2^{l(s+v)}} f|^q dt ds \right)^{1/q} \right\|_{L^q(\mathbb{R}^{n_1+n_2})}, \end{aligned}$$

and hence

$$\begin{aligned} & \|T_l^{u,v} f\|_{S_{q,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^q \\ & \leq C l^2 \int \int_{\mathbb{R}^2} \|2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} \sigma_{2^{lt}, 2^{ls}} * S_{2^{l(t+u)}, 2^{l(s+v)}} f\|_{L^q(\mathbb{R}^{n_1+n_2})}^q dt ds \\ & \leq C l^4 b_l^q \|h\|_{q'}^q \int \int_{\mathbb{R}^{n_1+n_2}} \int \int_{\mathbb{R}^2} |2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} S_{2^{l(t+u)}, 2^{l(s+v)}} f|^q dt ds dx_1 dx_2 \\ & = C l^2 b_l^q \|h\|_{q'}^q \int \int_{\mathbb{R}^{n_1+n_2}} \int_0^\infty \int_0^\infty |t_1^{-\alpha_1} t_2^{-\alpha_2} \Psi_{t_1, t_2} * f(x_1, x_2)|^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} dx_1 dx_2, \end{aligned}$$

where the last inequality follows from inequality (3.18) of Lemma 3. Therefore,

$$\|T_l^{u,v} f\|_{S_{q,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C b_l l^{2/q} \|h\|_{q'} \|f\|_{S_{q,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}. \tag{3.20}$$

When $p > q$, there exists a function $g \in L^{r'}(\mathbb{R}^{n_1+n_2})$ ($r = p/q$), $g \geq 0$, with unit norm such that

$$\begin{aligned}
 & \|T_l^{u,v} f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^q \\
 & \leq C l^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^{n_1+n_2}} |2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} \sigma_{2^{lt}, 2^{ls}} * S_{2^{l(t+u)}, 2^{l(s+v)}} f(x)|^q g(x) dx dt ds \\
 & \leq C l^2 b_l^{q/q'} \|h\|_{q'}^q \int_{\mathbb{R}^2} \int_{\mathbb{R}^{n_1+n_2}} 2^{-q\alpha_1 l(t+u)} 2^{-q\alpha_2 l(s+v)} L_{t,s}(|S_{2^{l(t+u)}, 2^{l(s+v)}} f|^q)(x) g(x) dx dt ds \\
 & = C l^2 b_l^{q/q'} \|h\|_{q'}^q \int_{\mathbb{R}^2} \int_{\mathbb{R}^{n_1+n_2}} |2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} S_{2^{l(t+u)}, 2^{l(s+v)}} f(x)|^q \tilde{L}_{t,s} g(x) dx dt ds \\
 & \leq C l^2 b_l^{q/q'} \|h\|_{q'}^q \int_{\mathbb{R}^{n_1+n_2}} \int_{\mathbb{R}^2} |2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} S_{2^{l(t+u)}, 2^{l(s+v)}} f(x)|^q dt ds \sup_{t,s \in \mathbb{R}} \tilde{L}_{t,s} g(x) dx \\
 & \leq C l^2 b_l^{q/q'} \|h\|_{q'}^q \left(\int_{\mathbb{R}^{n_1+n_2}} \left(\int_{\mathbb{R}^2} |2^{-\alpha_1 l(t+u)} 2^{-\alpha_2 l(s+v)} S_{2^{l(t+u)}, 2^{l(s+v)}} f(x)|^q dt ds \right)^r dx \right)^{1/r} \\
 & \quad \times \left(\int_{\mathbb{R}^{n_1+n_2}} \left| \sup_{t,s \in \mathbb{R}} \tilde{L}_{t,s} g(x) \right|^{r'} dx \right)^{1/r'} \\
 & \leq C l^2 b_l^{1+q/q'} \|h\|_{q'}^q \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^q \|g\|_{L^{r'}(\mathbb{R}^{n_1+n_2})} \\
 & \leq C l^2 b_l^{1+q/q'} \|h\|_{q'}^q \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}^q.
 \end{aligned}$$

The second inequality above follows from inequality (3.17) of Lemma 3; and the next to last inequality follows from (3.19) of Lemma 3. Therefore,

$$\|T_l^{u,v} f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C b_l l^{2/q} \|h\|_{q'} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \text{ for } 1 < q < p < \infty, \tag{3.21}$$

which together with inequality (3.20) imply that

$$\|T_l^{u,v} f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C b_l l^{2/q} \|h\|_{q'} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \text{ for } 1 < q \leq p < \infty. \tag{3.22}$$

Recall that if $h \in \mathcal{H}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+)$, then $\|h\|_{q'} \leq \|h\|_{\mathcal{H}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+)} \leq C < \infty$. On the other hand, if $h \in \mathcal{C}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+)$, then $\|h\|_{q'} \leq 4l^{2/q'} \|h\|_{\mathcal{C}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+)} \leq C l^{2/q'}$. Now set $q = 2$ in (3.22) and by duality, we obtain

$$\|T_l^{u,v} f\|_{S_{p,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C b_l l^\nu \|f\|_{S_{p,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \text{ for } 1 < p < \infty, \tag{3.23}$$

where $\nu = 2$ if $h \in \mathcal{C}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+)$, otherwise $\nu = 1$ if $h \in \mathcal{H}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+)$. Interpolating (3.23) with (3.16) yields for $1 < p < \infty$,

$$\begin{aligned}
 & \|T_l^{u,v} f\|_{S_{p,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\
 & \leq \begin{cases} C_q b_l l^\nu 2^{-l(u-1)\theta_1} 2^{-l(v-1)\theta_1} \|f\|_{S_{p,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} & \text{for } 0 < \theta_1 \leq 1, u, v \geq 1, \\ C_q b_l l^\nu 2^{-l(u-1)\theta_2} 2^{\frac{v\theta_2}{4\delta_q}} \|f\|_{S_{p,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} & \text{for } 0 < \theta_2 \leq 1, u \geq 1, v < 1, \\ C_q b_l l^\nu 2^{\frac{u\theta_3}{4\delta_q}} 2^{-l(v-1)\theta_3} \|f\|_{S_{p,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} & \text{for } 0 < \theta_3 \leq 1, u < 1, v \geq 1, \\ C_q b_l l^\nu 2^{\frac{u\theta_4}{4\delta_q}} 2^{\frac{v\theta_4}{4\delta_q}} \|f\|_{S_{p,2}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} & \text{for } 0 < \theta_4 \leq 1, u, v < 1. \end{cases}
 \end{aligned} \tag{3.24}$$

Interpolating (3.22) with (3.24) yields for $1 < q \leq p < \infty$,

$$\|T_l^{u,v} f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq \begin{cases} C_q b_l l^{\nu_q} 2^{-l(u-1)\beta_1} 2^{-l(v-1)\beta_1} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} & \text{for } 0 < \beta_1 \leq \theta_1 \leq 1, u, v \geq 1, \\ C_q b_l l^{\nu_q} 2^{-l(u-1)\beta_2} 2^{\frac{v\beta_2}{4\delta q}} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} & \text{for } 0 < \beta_2 \leq \theta_2 \leq 1, u \geq 1, v < 1, \\ C_q b_l l^{\nu_q} 2^{\frac{u\beta_3}{4\delta q}} 2^{-l(v-1)\beta_3} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} & \text{for } 0 < \beta_3 \leq \theta_3 \leq 1, u < 1, v \geq 1, \\ C_q b_l l^{\nu_q} 2^{\frac{u\beta_4}{4\delta q}} 2^{\frac{v\beta_4}{4\delta q}} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} & \text{for } 0 < \beta_4 \leq \theta_4 \leq 1, u, v < 1, \end{cases} \tag{3.25}$$

where

$$\nu_q = \begin{cases} 2 & \text{if } h \in \mathcal{C}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+), \\ 1 & \text{if } h \in \mathcal{H}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+), \text{ and } q > 2, \\ 2/q & \text{if } h \in \mathcal{H}_{q'}(\mathbb{R}^+ \times \mathbb{R}^+), \text{ and } 1 < q \leq 2. \end{cases}$$

Inequalities (3.13) and (3.25) imply that

$$\|T_l f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C_{p,q} l^{\nu_q} \|\Omega\|_{L^1(E_l)} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \text{ for } 1 < q \leq p < \infty. \tag{3.26}$$

It remains to prove the inequality (3.3) for $1 < p < q < \infty$. We define the truncated singular integral $T_l^{\epsilon_1, \epsilon_2} f$ by

$$\begin{aligned} T_l^{\epsilon_1, \epsilon_2} f(x_1, x_2) &= \int_{|y_2| > \epsilon_2} \int_{|y_1| > \epsilon_1} \frac{h(|y_1|, |y_2|) \Omega_l(y'_1, y'_2)}{|y_1|^{n_1} |y_2|^{n_2}} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2 \\ &\equiv \int_{\mathbb{R}^{n_1+n_2}} \frac{h^{\epsilon_1, \epsilon_2}(|y_1|, |y_2|) \Omega_l(y'_1, y'_2)}{|y_1|^{n_1} |y_2|^{n_2}} f(x_1 - y_1, x_2 - y_2) dy_1 dy_2, \end{aligned}$$

where $h^{\epsilon_1, \epsilon_2}(|y_1|, |y_2|) = h(|y_1|, |y_2|) \chi_{\epsilon_1}(|y_1|) \chi_{\epsilon_2}(|y_2|)$, and $\chi_{\epsilon_i}(|y_i|)$ ($i = 1, 2$) is the characteristic function defined on the set $\{y_i \in \mathbb{R}^{n_i} : |y_i| > \epsilon_i > 0\}$. Note that for $1 < q < \infty$, $\|h^{\epsilon_1, \epsilon_2}\|_{\mathcal{A}_q} \leq \|h\|_{\mathcal{A}_q} \leq C < \infty$, where \mathcal{A}_q is either \mathcal{C}_q or \mathcal{H}_q , and the bound C is independent of $\epsilon_1, \epsilon_2 > 0$. Thus it follows that inequality (3.26) also holds for the truncated singular integral $T_l^{\epsilon_1, \epsilon_2} f$, and the bound $C_{p,q}$ is independent of $\epsilon_1, \epsilon_2 > 0$. By duality, we get

$$\|T_l^{\epsilon_1, \epsilon_2} f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C_{p,q} l^{\nu_q} \|\Omega\|_{L^1(E_l)} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \text{ for } 1 < p < q < \infty.$$

Letting $\epsilon_1, \epsilon_2 \rightarrow 0$, we obtain

$$\|T_l f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C_{p,q} l^{\nu_q} \|\Omega\|_{L^1(E_l)} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \text{ for } 1 < p < q < \infty,$$

which together with inequality (3.26) imply that

$$\|T_l f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq C_{p,q} l^{\nu_q} \|\Omega\|_{L^1(E_l)} \|f\|_{S_{p,q}^{\alpha_1, \alpha_2} \dot{F}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \text{ for } 1 < p, q < \infty.$$

The proof of Theorem 1 is finished. □

Remark Since the technique used in the proof of Theorem 1 also works for one-parameter case, we will state below the result for the the one-parameter case without its proof.

Theorem 2 Let h be a measurable function defined on \mathbb{R}^+ . Let Ω be a function defined on S^{n-1} ($n \geq 2$), which is homogeneous of degree zero and satisfies the usual cancellation condition. For $f \in \mathcal{S}(\mathbb{R}^n)$, define the singular integral Tf by

$$Tf(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{\Omega(y') h(|y|)}{|y|^n} f(x - y) dy.$$

Set

$$\alpha_q = \begin{cases} 1 & \text{if } h \in \mathcal{C}_{q'}(\mathbb{R}^+), \\ 1/2 & \text{if } h \in \mathcal{H}_{q'}(\mathbb{R}^+), \text{ and } q > 2, \\ 1/q & \text{if } h \in \mathcal{H}_{q'}(\mathbb{R}^+), \text{ and } 1 < q \leq 2. \end{cases}$$

If $\Omega \in L(\log L^+)^{\alpha_q}(S^{n-1})$, then for any $p, q \in (1, \infty)$, $s \in \mathbb{R}$, T has a bounded extension on $\dot{F}_p^{s,q}(\mathbb{R}^n)$. That is,

$$\|Tf\|_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \leq C_{p,q} \|\Omega\|_{L(\log L^+)^{\alpha_q}(S^{n-1})} \|f\|_{\dot{F}_p^{s,q}(\mathbb{R}^n)} \text{ for } 1 < p, q < \infty, s \in \mathbb{R}.$$

References

- [1] Al-Salman A, Al-Jarrah A A. On rough singular integrals along surfaces on product domains. Turk J Math, 2005, **29**: 275–285
- [2] Al-Salman A, Al-Qassem H, Pan Yibiao. Singular integrals on product domains. Indiana Univ Math J, 2006, **55**(1): 369–387
- [3] Al-Salman A, Al-Qassem H, Cheng L C, Pan Y. L^p bounds for the function of Marcinkiewicz. Math Res Lett, 2002, **9**: 697–700
- [4] Al-Qassem H, Pan Yibiao. L^p boundedness for singular integrals with rough kernels on product domains. Hokkaido Mathe J, 2002, **31**: 555–613
- [5] Al-Qassem H, Ali M. L^p boundedness for singular integral operators with $L(\log L^+)^2$ kernels on product spaces. Kyungpook Math J, 2006, **46**: 377–387
- [6] Chen Jiecheng, Fan Dashan, Ying Yiming. Singular integral operators on function spaces. J Math Anal Appl, 2002, **276**(2): 691–708
- [7] Chen Larry, Le H V. Singular integrals with mixed homogeneity in product spaces. Math Ineq Appl, 2011, **14**(1): 155–172
- [8] Duoandikoetxea J. Multiple singular integrals and maximal functions along hypersurfaces. Annales de l'institut Fourier, 1986, **36**(4): 185–206
- [9] Duoandikoetxea J, de Francia J L R. Maximal and singular integral operators via Fourier transform estimates. Invent Math, 1986, **84**(3): 541–561
- [10] Fan Dashan, Wu Huoxiong. On the generalized Marcinkiewicz integral operators with rough kernels. Canad Math Bull, 2011, **54**(1): 100–112
- [11] Fefferman R. Singular integrals on product spaces. Adv Math, 1982, **45**: 117–143
- [12] Schmeisser H -J. Recent developments in the theory of function spaces with dominating mixed smoothness//Nonlinear Analysis, Function Spaces and Applications. Proceedings of the Spring School Held in Prague, May 30-June 6, 2006. Vol 8. Czech Academy of Sciences, Mathematical Institute, Praha, 2007: 145–204.
- [13] Schmeisser H -J, Triebel H. Topics in Fourier Analysis and Function Spaces. Wiley: Chister, 1987
- [14] Triebel H. Theory of Function Spaces, Monographs in Mathematics, **78**. Basel: Birkhäuser Verlag, 1983
- [15] Walsh T. On the function of Marcinkiewicz. Studia Math, 1972, **78**: 203–217