Robert J. Elliott and P. Ekkehard Kopp

Mathematics of Financial Markets

Second edition

Springer

Preface

This work is aimed at an audience with a sound mathematical background wishing to learn about the rapidly expanding field of mathematical finance. Its content is suitable particularly for graduate students in mathematics who have a background in measure theory and probability.

The emphasis throughout is on developing the mathematical concepts required for the theory within the context of their application. No attempt is made to cover the bewildering variety of novel (or 'exotic') financial instruments that now appear on the derivatives markets; the focus throughout remains on a rigorous development of the more basic options that lie at the heart of the remarkable range of current applications of martingale theory to financial markets.

The first five chapters present the theory in a discrete-time framework. Stochastic calculus is not required, and this material should be accessible to anyone familiar with elementary probability theory and linear algebra.

The basic idea of pricing by arbitrage (or, rather, by non-arbitrage) is presented in Chapter 1. The unique price for a European option in a single-period binomial model is given and then extended to multi-period binomial models. Chapter 2 introduces the idea of a martingale measure for price processes. Following a discussion of the use of self-financing trading strategies to hedge against trading risk, it is shown how options can be priced using an equivalent measure for which the discounted price process is a martingale. This is illustrated for the simple binomial Cox-Ross-Rubinstein pricing models, and the Black-Scholes formula is derived as the limit of the prices obtained for such models. Chapter 3 gives the 'fundamental theorem of asset pricing', which states that if the market does not contain arbitrage opportunities there is an equivalent martingale measure. Explicit constructions of such measures are given in the setting of finite market models. Completeness of markets is investigated in Chapter 4; in a complete market, every contingent claim can be generated by an admissible self-financing strategy (and the martingale measure is unique). Stopping times, martingale convergence results, and American options are discussed in a discrete-time framework in Chapter 5.

The second five chapters of the book give the theory in continuous time. This begins in Chapter 6 with a review of the stochastic calculus. Stopping times, Brownian motion, stochastic integrals, and the Itô differentiation rule are all defined and discussed, and properties of stochastic differential equations developed.

The continuous-time pricing of European options is developed in Chapter 7. Girsanov's theorem and martingale representation results are developed, and the Black-Scholes formula derived. Optimal stopping results are applied in Chapter 8 to a thorough study of the pricing of American options, particularly the American put option.

Chapter 9 considers selected results on term structure models, forward and future prices, and change of numéraire, while Chapter 10 presents the basic framework for the study of investment and consumption problems.

Acknowledgments Sections of the book have been presented in courses at the Universities of Adelaide and Alberta. The text has consequently benefited from subsequent comments and criticism. Our particular thanks go to Monique Jeanblanc-Piqué, whose careful reading of the text and valuable comments led to many improvements. Many thanks are also due to Volker Wellmann for reading much of the text and for his patient work in producing consistent TEX files and the illustrations.

Finally, the authors wish to express their sincere thanks to the Social Sciences and Humanities Research Council of Canada for its financial support of this project.

Edmonton, Alberta, Canada — Robert J. Elliott
Hull, United Kingdom — P. Ekkehard Kopp

Preface to the Second Edition

This second, revised edition contains a significant number of changes and additions to the original text. We were guided in our choices by the comments of a number of readers and reviewers as well as instructors using the text with graduate classes, and we are grateful to them for their advice. Any errors that remain are of course entirely our responsibility.

In the five years since the book was first published, the subject has continued to grow at an astonishing rate. Graduate courses in mathematical finance have expanded from their business school origins to become standard fare in many mathematics departments in Europe and North America and are spreading rapidly elsewhere, attracting large numbers of students. Texts for this market have multiplied, as the rapid growth of the *Springer Finance* series testifies. In choosing new material, we have therefore focused on topics that aid the student's understanding of the fundamental concepts, while ensuring that the techniques and ideas presented remain up to date. We have given particular attention, in part through revisions to Chapters 5 and 6, to linking key ideas occurring in the two main sections (discrete- and continuous-time derivatives) more closely and explicitly.

Chapter 1 has been revised to include a discussion of risk and return in the one-step binomial model (which is given a new, extended presentation) and this is complemented by a similar treatment of the Black-Scholes model in Chapter 7. Discussion of elementary bounds for option prices in Chapter 1 is linked to sensitivity analysis of the Black-Scholes price (the 'Greeks') in Chapter 7, and call-put parity is utilised in various settings.

Chapter 2 includes new sections on superhedging and the use of extended trading strategies that include contingent claims, as well as a more elegant derivation of the Black-Scholes option price as a limit of binomial approximants.

Chapter 3 includes a substantial new section leading to a complete proof of the equivalence, for discrete-time models, of the no-arbitrage condition and the existence of equivalent martingale measures. The proof, while not original, is hopefully more accessible than others in the literature.

This material leads in Chapter 4 to a characterisation of the arbitrage interval for general market models and thus to a characterisation of complete models, showing in particular that complete models must be finitely generated.

The new edition ends with a new chapter on risk measures, a subject that has become a major area of research in the past five years. We include a brief introduction to Value at Risk and give reasons why the use of coherent risk measures (or their more recent variant, deviation measures) is to be preferred. Chapter 11 ends with an outline of the use of risk measures in recent work on partial hedging of contingent claims.

The changes we have made to the text have been informed by our continuing experience in teaching graduate courses at the universities of Adelaide, Calgary and Hull, and at the African Institute for Mathematical Sciences in Cape Town.

Acknowledgments Particular thanks are due to Alet Roux (Hull) and Andrew Royal (Calgary) who provided invaluable assistance with the complexities of LaTeX typesetting and who read large sections of the text. Thanks are also due to the Social Sciences and Humanities Research Council of Canada for continuing financial support.

Calgary, Alberta, Canada — Robert J. Elliott
Hull, United Kingdom — P. Ekkehard Kopp
May 2004

Contents