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# Introduction to Partial Differential Equations

A Computational Approach

With 69 Illustrations

 Springer

# Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)*.

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

*TAM* will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research-level monographs.

# Preface

*“It is impossible to exaggerate the extent to which modern applied mathematics has been shaped and fueled by the general availability of fast computers with large memories. Their impact on mathematics, both applied and pure, is comparable to the role of the telescopes in astronomy and microscopes in biology.”*

— *Peter Lax, Siam Rev. Vol. 31 No. 4*

Congratulations! You have chosen to study partial differential equations. That decision is a wise one; the laws of nature are written in the language of partial differential equations. Therefore, these equations arise as models in virtually all branches of science and technology. Our goal in this book is to help you to understand what this vast subject is about. The book is an introduction to the field. We assume only that you are familiar with basic calculus and elementary linear algebra. Some experience with ordinary differential equations would also be an advantage.

Introductory courses in partial differential equations are given all over the world in various forms. The traditional approach to the subject is to introduce a number of analytical techniques, enabling the student to derive exact solutions of some simplified problems. Students who learn about

computational techniques on other courses subsequently realize the scope of partial differential equations beyond paper and pencil.

Our approach is different. We introduce analytical and computational techniques in the same book and thus in the same course. The main reason for doing this is that the computer, developed to assist scientists in solving partial differential equations, has become commonly available and is currently used in all practical applications of partial differential equations. Therefore, a modern introduction to this topic must focus on methods suitable for computers. But these methods often rely on deep analytical insight into the equations. We must therefore take great care not to throw away basic analytical methods but seek a sound balance between analytical and computational techniques.

One advantage of introducing computational techniques is that nonlinear problems can be given more attention than is common in a purely analytical introduction. We have included several examples of nonlinear equations in addition to the standard linear models which are present in any introductory text. In particular we have included a discussion of reaction-diffusion equations. The reason for this is their widespread application as important models in various scientific applications.

Our aim is not to discuss the merits of different numerical techniques. There are a huge number of papers in scientific journals comparing different methods to solve various problems. We do not want to include such discussions. Our aim is to demonstrate that computational techniques are simple to use and often give very nice results, not to show that even better results can be obtained if slightly different methods are used. We touch briefly upon some such discussion, but not in any major way, since this really belongs to the field of numerical analysis and should be taught in separate courses. Having said this, we always try to use the simplest possible numerical techniques. This should in no way be interpreted as an attempt to advocate certain methods as opposed to others; they are merely chosen for their simplicity.

Simplicity is also our reason for choosing to present exclusively finite difference techniques. The entire text could just as well be based on finite element techniques, which definitely have greater potential from an application point of view but are slightly harder to understand than their finite difference counterparts.

We have attempted to present the material at an easy pace, explaining carefully both the ideas and details of the derivations. This is particularly the case in the first chapters but subsequently less details are included and some steps are left for the reader to fill in. There are a lot of exercises included, ranging from the straightforward to more challenging ones. Some of them include a bit of implementation and some experiments to be done on the computer. We strongly encourage students not to skip these parts. In addition there are some "projects." These are either included to refresh

the student's memory of results needed in this course, or to extend the theories developed in the present text.

Given the fact that we introduce both numerical and analytical tools, we have chosen to put little emphasis on modeling. Certainly, the derivation of models based on partial differential equations is an important topic, but it is also very large and can therefore not be covered in detail here.

The first seven chapters of this book contain an elementary course in partial differential equations. Topics like separation of variables, energy arguments, maximum principles, and finite difference methods are discussed for the three basic linear partial differential equations, i.e. the heat equation, the wave equation, and Poisson's equation. In Chapters 8–10 more theoretical questions related to separation of variables and convergence of Fourier series are discussed. The purpose of Chapter 11 is to introduce nonlinear partial differential equations. In particular, we want to illustrate how easily finite difference methods adapt to such problems, even if these equations may be hard to handle by an analytical approach. In Chapter 12 we give a brief introduction to the Fourier transform and its application to partial differential equations.

Some of the exercises in this text are small computer projects involving a bit of programming. This programming could be done in any language. In order to get started with these projects, you may find it useful to pick up some examples from our web site, <http://www.ifl.uio.no/~pde/>, where you will find some Matlab code and some simple Java applets.

## Acknowledgments

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# Contents

<b>1</b>	<b>Setting the Scene</b>	<b>1</b>
1.1	What Is a Differential Equation? . . . . .	1
1.1.1	Concepts . . . . .	2
1.2	The Solution and Its Properties . . . . .	4
1.2.1	An Ordinary Differential Equation . . . . .	4
1.3	A Numerical Method . . . . .	6
1.4	Cauchy Problems . . . . .	10
1.4.1	First-Order Homogeneous Equations . . . . .	11
1.4.2	First-Order Nonhomogeneous Equations . . . . .	14
1.4.3	The Wave Equation . . . . .	15
1.4.4	The Heat Equation . . . . .	18
1.5	Exercises . . . . .	20
1.6	Projects . . . . .	28
<b>2</b>	<b>Two-Point Boundary Value Problems</b>	<b>39</b>
2.1	Poisson's Equation in One Dimension . . . . .	40
2.1.1	Green's Function . . . . .	42
2.1.2	Smoothness of the Solution . . . . .	43
2.1.3	A Maximum Principle . . . . .	44
2.2	A Finite Difference Approximation . . . . .	45
2.2.1	Taylor Series . . . . .	46
2.2.2	A System of Algebraic Equations . . . . .	47
2.2.3	Gaussian Elimination for Tridiagonal Linear Systems	50
2.2.4	Diagonal Dominant Matrices . . . . .	53

2.2.5	Positive Definite Matrices . . . . .	55
2.3	Continuous and Discrete Solutions . . . . .	57
2.3.1	Difference and Differential Equations . . . . .	57
2.3.2	Symmetry . . . . .	58
2.3.3	Uniqueness . . . . .	61
2.3.4	A Maximum Principle for the Discrete Problem . . . . .	61
2.3.5	Convergence of the Discrete Solutions . . . . .	63
2.4	Eigenvalue Problems . . . . .	65
2.4.1	The Continuous Eigenvalue Problem . . . . .	65
2.4.2	The Discrete Eigenvalue Problem . . . . .	68
2.5	Exercises . . . . .	72
2.6	Projects . . . . .	82
<b>3</b>	<b>The Heat Equation</b> . . . . .	<b>87</b>
3.1	A Brief Overview . . . . .	88
3.2	Separation of Variables . . . . .	90
3.3	The Principle of Superposition . . . . .	92
3.4	Fourier Coefficients . . . . .	95
3.5	Other Boundary Conditions . . . . .	97
3.6	The Neumann Problem . . . . .	98
3.6.1	The Eigenvalue Problem . . . . .	99
3.6.2	Particular Solutions . . . . .	100
3.6.3	A Formal Solution . . . . .	101
3.7	Energy Arguments . . . . .	102
3.8	Differentiation of Integrals . . . . .	106
3.9	Exercises . . . . .	108
3.10	Projects . . . . .	113
<b>4</b>	<b>Finite Difference Schemes for the Heat Equation</b> . . . . .	<b>117</b>
4.1	An Explicit Scheme . . . . .	119
4.2	Fourier Analysis of the Numerical Solution . . . . .	122
4.2.1	Particular Solutions . . . . .	123
4.2.2	Comparison of the Analytical and Discrete Solution . . . . .	127
4.2.3	Stability Considerations . . . . .	129
4.2.4	The Accuracy of the Approximation . . . . .	130
4.2.5	Summary of the Comparison . . . . .	131
4.3	Von Neumann's Stability Analysis . . . . .	132
4.3.1	Particular Solutions: Continuous and Discrete . . . . .	133
4.3.2	Examples . . . . .	134
4.3.3	A Nonlinear Problem . . . . .	137
4.4	An Implicit Scheme . . . . .	140
4.4.1	Stability Analysis . . . . .	143
4.5	Numerical Stability by Energy Arguments . . . . .	145
4.6	Exercises . . . . .	148

<b>5</b>	<b>The Wave Equation</b>	<b>159</b>
5.1	Separation of Variables . . . . .	160
5.2	Uniqueness and Energy Arguments . . . . .	163
5.3	A Finite Difference Approximation . . . . .	165
5.3.1	Stability Analysis . . . . .	168
5.4	Exercises . . . . .	170
<b>6</b>	<b>Maximum Principles</b>	<b>175</b>
6.1	A Two-Point Boundary Value Problem . . . . .	175
6.2	The Linear Heat Equation . . . . .	178
6.2.1	The Continuous Case . . . . .	180
6.2.2	Uniqueness and Stability . . . . .	183
6.2.3	The Explicit Finite Difference Scheme . . . . .	184
6.2.4	The Implicit Finite Difference Scheme . . . . .	186
6.3	The Nonlinear Heat Equation . . . . .	188
6.3.1	The Continuous Case . . . . .	189
6.3.2	An Explicit Finite Difference Scheme . . . . .	190
6.4	Harmonic Functions . . . . .	191
6.4.1	Maximum Principles for Harmonic Functions . . . . .	193
6.5	Discrete Harmonic Functions . . . . .	195
6.6	Exercises . . . . .	201
<b>7</b>	<b>Poisson's Equation in Two Space Dimensions</b>	<b>209</b>
7.1	Rectangular Domains . . . . .	209
7.2	Polar Coordinates . . . . .	212
7.2.1	The Disc . . . . .	213
7.2.2	A Wedge . . . . .	216
7.2.3	A Corner Singularity . . . . .	217
7.3	Applications of the Divergence Theorem . . . . .	218
7.4	The Mean Value Property for Harmonic Functions . . . . .	222
7.5	A Finite Difference Approximation . . . . .	225
7.5.1	The Five-Point Stencil . . . . .	225
7.5.2	An Error Estimate . . . . .	228
7.6	Gaussian Elimination for General Systems . . . . .	230
7.6.1	Upper Triangular Systems . . . . .	230
7.6.2	General Systems . . . . .	231
7.6.3	Banded Systems . . . . .	234
7.6.4	Positive Definite Systems . . . . .	236
7.7	Exercises . . . . .	237
<b>8</b>	<b>Orthogonality and General Fourier Series</b>	<b>245</b>
8.1	The Full Fourier Series . . . . .	246
8.1.1	Even and Odd Functions . . . . .	249
8.1.2	Differentiation of Fourier Series . . . . .	252
8.1.3	The Complex Form . . . . .	255



8.1.4	Changing the Scale . . . . .	256
8.2	Boundary Value Problems and Orthogonal Functions . . . . .	257
8.2.1	Other Boundary Conditions . . . . .	257
8.2.2	Sturm-Liouville Problems . . . . .	261
8.3	The Mean Square Distance . . . . .	264
8.4	General Fourier Series . . . . .	267
8.5	A Poincaré Inequality . . . . .	273
8.6	Exercises . . . . .	276
<b>9</b>	<b>Convergence of Fourier Series</b>	<b>285</b>
9.1	Different Notions of Convergence . . . . .	285
9.2	Pointwise Convergence . . . . .	290
9.3	Uniform Convergence . . . . .	296
9.4	Mean Square Convergence . . . . .	300
9.5	Smoothness and Decay of Fourier Coefficients . . . . .	302
9.6	Exercises . . . . .	307
<b>10</b>	<b>The Heat Equation Revisited</b>	<b>313</b>
10.1	Compatibility Conditions . . . . .	314
10.2	Fourier's Method: A Mathematical Justification . . . . .	319
10.2.1	The Smoothing Property . . . . .	319
10.2.2	The Differential Equation . . . . .	321
10.2.3	The Initial Condition . . . . .	323
10.2.4	Smooth and Compatible Initial Functions . . . . .	325
10.3	Convergence of Finite Difference Solutions . . . . .	327
10.4	Exercises . . . . .	331
<b>11</b>	<b>Reaction-Diffusion Equations</b>	<b>337</b>
11.1	The Logistic Model of Population Growth . . . . .	337
11.1.1	A Numerical Method for the Logistic Model . . . . .	339
11.2	Fisher's Equation . . . . .	340
11.3	A Finite Difference Scheme for Fisher's Equation . . . . .	342
11.4	An Invariant Region . . . . .	343
11.5	The Asymptotic Solution . . . . .	346
11.6	Energy Arguments . . . . .	349
11.6.1	An Invariant Region . . . . .	350
11.6.2	Convergence Towards Equilibrium . . . . .	351
11.6.3	Decay of Derivatives . . . . .	352
11.7	Blowup of Solutions . . . . .	354
11.8	Exercises . . . . .	357
11.9	Projects . . . . .	360
<b>12</b>	<b>Applications of the Fourier Transform</b>	<b>365</b>
12.1	The Fourier Transform . . . . .	366
12.2	Properties of the Fourier Transform . . . . .	368

12.3	The Inversion Formula . . . . .	372
12.4	The Convolution . . . . .	375
12.5	Partial Differential Equations . . . . .	377
12.5.1	The Heat Equation . . . . .	377
12.5.2	Laplace's Equation in a Half-Plane . . . . .	380
12.6	Exercises . . . . .	382
	<b>References</b>	<b>385</b>
	<b>Index</b>	<b>389</b>