Elementary Linear Algebra

Fifth Edition

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Preface for the Instructor

This textbook is intended for a sophomore- or junior-level introductory course in linear algebra. We assume the students have had at least one course in calculus.

PHILOSOPHY OF THE TEXT

Helpful Transition from Computation to Theory: Our main objective in writing this textbook was to present the basic concepts of linear algebra as clearly as possible. The "heart" of this text is the material in Chapters 4 and 5 (vector spaces, linear transformations). In particular, we have taken special care to guide students through these chapters as the emphasis changes from computation to abstract theory. Many theoretical concepts (such as linear combinations of vectors, the row space of a matrix, and eigenvalues and eigenvectors) are first introduced in the early chapters in order to facilitate a smoother transition to later chapters. Please encourage the students to read the text deeply and thoroughly.

Applications of Linear Algebra and Numerical Techniques: This text contains a wide variety of applications of linear algebra, as well as all of the standard numerical techniques typically found in most introductory linear algebra texts. Aside from the many applications and techniques already presented in the first seven chapters, Chapter 8 is devoted entirely to additional applications, while Chapter 9 introduces several other numerical techniques. A summary of these applications and techniques is given in the chart located at the end of the Prefaces.

Numerous Examples and Exercises: There are 340 numbered examples in the text, at least one for each major concept or application, as well as for almost every theorem. The text also contains an unusually large number of exercises. There are more than 970 numbered exercises, and many of these have multiple parts, for a total of more than 2600 questions. The exercises within each section are generally ordered by increasing difficulty, beginning with basic computational problems and moving on to more theoretical problems and proofs. Answers are provided at the end of the book for approximately half of the computational exercises; these problems are marked with a star (⋆). Full solutions to these starred exercises appear in the Student Solutions Manual. The last exercises in each section are True/False questions (there are over 500 of these altogether). These are designed to test the students' understanding of fundamental concepts by emphasizing the importance of critical words in definitions or theorems. Finally, there is a set of comprehensive Review Exercises at the end of each of Chapters 1 through 7.

Assistance in the Reading and Writing of Mathematical Proofs: To prepare students for an increasing emphasis on abstract concepts, we introduce them to proof-reading and proof-writing very early in the text, beginning with Section 1.3, which is devoted solely to this topic. For long proofs, we present students with an overview so they do not get lost in the details. For every *nontrivial* theorem in Chapters 1 through 6, we have either included a proof, or given detailed hints to enable students to provide a proof on their own. Most of the proofs left as exercises are marked with the symbol ▶, and these proofs can be found in the Student Solutions Manual.

Symbol Table: Following the Prefaces, for convenience, there is a comprehensive Symbol Table summarizing all of the major symbols for linear algebra that are employed in this text.

Instructor's Manual: An Instructor's Manual is available online for all instructors who adopt this text. This manual contains the answers to all exercises, both computational and theoretical. This manual also includes three versions of a Sample Test for each of Chapters 1 through 7, along with corresponding answer keys.

Student Solutions Manual: A Student Solutions Manual is available for students to purchase online. This manual contains full solutions for each exercise in the text bearing a ⋆ (those whose answers appear in Appendix E). The Student Solutions Manual also contains the proofs of most of the theorems that were left to the exercises. These exercises are marked in the text with a ▶. Because we have compiled this manual ourselves, it utilizes the same styles of proof-writing and solution techniques that appear in the actual text.

Additional Material on the Web: The web site for this textbook is:

http://store.elsevier.com/9780128008539

This site contains information about the text, as well as several sections of subject content that are available for instructors and students who adopt the text. These web sections range from elementary to advanced topics – from the Cross Product to Jordan Canonical Form. These can be covered by instructors in the classroom, or used by the students to explore on their own.

MAJOR CHANGES FOR THE FIFTH EDITION

We list some of the most important specific changes section by section here, but for brevity's sake, we will not detail every specific change. However, some of the more general systemic revisions include:

- Some reordering of theorems and examples was done in various chapters (particularly in Chapters 2 through 5) to streamline the presentation of results (see, e.g., Theorem 2.2 and Corollary 2.3, Theorem 4.9, Theorem 5.16, Corollary 6.16, Theorem 6.21, Corollary 6.23, and Corollary 7.21).

- Various important, frequently used, statements in the text have been converted into formal theorems for easier reference in proofs and solutions to exercises (e.g., in Sections 6.2, 7.5, and 8.8).
- The Highlights appearing at the conclusion of each section have been revised substantially to be more concrete in nature so that they are more useful to students looking for a quick review of the section.
- Many formatting changes were made in the textbook for greater readability to display some intermediate steps and final results of solutions more prominently.

Other sections that received substantial changes from the fourth edition are:

- **Section 1.1** (and beyond)**:** The notion of a "Generalized Etch A Sketch®"[1] has been introduced, beginning in Section 1.1, to help students better understand the vector concepts of linear combinations, span, and linear independence.
- **Section 3.3:** The text now includes a proof for (the current) Theorem 3.10, eliminating the need for a long sequence of intertwined exercises that formerly concluded this section. Also, the material on the classical adjoint of a matrix has been moved to the exercises.
- **Section 4.5:** The proof of (the current) Lemma 4.10 has been shortened. In order to sharpen the focus on more essential concepts, the material on maximal linearly independent subsets and minimal spanning subsets has been eliminated. This necessitated a change in the proof that a subspace of a finite-dimensional vector space is finite dimensional. Section 4.6 was also adjusted accordingly.
- **Section 4.6:** The Inspection Method (for reducing a finite spanning set to a basis) has been moved to the exercises.
- **Section 5.4:** The result now stated as Corollary 5.13 was moved here from Section 5.5.
- **Section 8.1:** This section on Graph Theory was thoroughly updated, with more general graphs and digraphs treated, rather than just simple graphs and digraphs. Also, new material has been added on the connectivity of graphs.
- **Section 8.3:** The concept of "interpolation" has been introduced.
- **Appendix B:** This appendix on basic properties of functions was thoroughly rewritten to make theorems and examples more visible.
- **Appendix C:** Statements of the commutative, associative, and distributive laws for addition and multiplication of complex numbers have been added.
- **Appendix D:** The section "Elementary Matrices" (formerly Section 8.6 in the fourth edition) has been moved to Appendix D. Consequently, the answers to starred exercises now appear in Appendix E.

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[1] Etch A Sketch® is a registered trademark of the Ohio Art Company.

- The **Front Tables** and **End Tables** inside the front and back covers of the book contain several additional Methods presented in the text.
- Finally, both the **Instructor's Manual** and **Student Solutions Manual** have been totally reformatted throughout. In particular, a substantially larger number of intermediate steps and final results of solutions are now displayed more prominently for greater readability.

PLANS FOR COVERAGE

Chapters 1 through 6 have been written in a sequential fashion. Each section is generally needed as a prerequisite for what follows. Therefore, we recommend that these sections be covered in order. However, Section 1.3 (An Introduction to Proofs) can be covered, in whole, or in part, any time after Section 1.2. Also, the material in Section 6.1 (Orthogonal Bases and the Gram-Schmidt Process) can be covered any time after Chapter 4.

The sections in Chapters 7 through 9 can be covered at any time as long as the prerequisites for those sections have previously been covered. (Consult the Prerequisite Chart below for the sections in Chapters 7, 8, and 9.)

The textbook contains much more material than can be covered in a typical 3- or 4-credit course. Some of the material in Chapter 1 could be reviewed quickly if students are already familiar with vector and matrix operations. Two suggested timetables for covering the material in this text are presented below — one for a 3-credit course, and the other for a 4-credit course. A 3-credit course could skip portions of Sections 1.3, 2.3, 3.3, 4.1, 5.5, 5.6, 6.2, and 6.3, and all of Chapter 7. A 4-credit course could cover most of the material of Chapters 1 through 6 (skipping some portions of Sections 1.3, 2.3, and 3.3), and also cover some of Chapter 7.

	3-Credit Course	4-Credit Course
Chapter 1	5 classes	5 classes
Chapter 2	5 classes	6 classes
Chapter 3	5 classes	5 classes
Chapter 4	11 classes	13 classes
Chapter 5	8 classes	13 classes
Chapter 6	2 classes	5 classes
Chapter 7		2 classes
Chapters 8 & 9 (selections)	3 classes	4 classes
Tests	3 classes	3 classes
Total	42 classes	56 classes

PREREQUISITE CHART FOR LATER SECTIONS

Prerequisites for the material in later sections of the text are listed in the following chart. While each section of Chapter 7 depends on the sections that precede it, the sections of Chapters 8 and 9 are generally independent of each other, and they can be covered as soon as their prerequisites from earlier chapters have been met. Also note that the techniques for solving differential equations in Section 8.8 require only Section 3.4 as a prerequisite, although terminology from Chapters 4 and 5 is used throughout Section 8.8.

Section	Prerequisite
Section 7.1 (Complex n–Vectors and Matrices)	Section 1.5
Section 7.2 (Complex Eigenvalues and Complex Eigenvectors)	Section 3.4
Section 7.3 (Complex Vector Spaces)	Section 5.2
Section 7.4 (Orthogonality in $\mathbb{C}^n$)	Section 6.3
Section 7.5 (Inner Product Spaces)	Section 6.3
Section 8.1 (Graph Theory)	Section 1.5
Section 8.2 (Ohm's Law)	Section 2.2
Section 8.3 (Least-Squares Polynomials)	Section 2.2
Section 8.4 (Markov Chains)	Section 2.2
Section 8.5 (Hill Substitution: An Intro. to Coding Theory)	Section 2.4
Section 8.6 (Rotation of Axes)	Section 4.7
Section 8.7 (Computer Graphics)	Section 5.2
Section 8.8 (Differential Equations)	Section 5.6
Section 8.9 (Least-Squares Solutions for Inconsistent Systems)	Section 6.2
Section 8.10 (Quadratic Forms)	Section 6.3
Section 9.1 (Numerical Methods for Solving Systems)	Section 2.3
Section 9.2 (**LDU** Decomposition)	Section 2.4
Section 9.3 (The Power Method for Finding Eigenvalues)	Section 3.4
Section 9.4 (**QR** Factorization)	Section 6.1
Section 9.5 (Singular Value Decomposition)	Section 6.3
Appendix D (Elementary Matrices)	Section 2.4

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We especially thank those students and instructors who have reviewed earlier editions of the textbook as well as those who have classroom-tested versions of the earlier editions of the manuscript. Their comments and suggestions have been extremely helpful, and have guided us in shaping the text in many ways.

Last, but most important of all, we want to thank our wives, Ene and Lyn, for bearing extra hardships so that we could work on this text. Their love and support continues to be an inspiration.

Preface to the Student

A Quick Overview of the Text: Chapters 1 to 3 present the basic tools for your study of linear algebra: vectors, matrices, systems of linear equations, inverses, determinants, and eigenvalues. Chapters 4 to 6 then treat these concepts on a higher level: vector spaces, spanning, linear independence, bases, coordinatization, linear transformations, kernel, range, isomorphisms, and orthogonality. Chapter 7 extends the results of earlier chapters to the complex number system. Chapters 8 and 9 present many applications and numerical techniques widely used in linear algebra.

Strategies for Learning: Many students find that the transition to abstractness (beginning with general vector spaces in Chapter 4) is challenging. This text was written specifically to help you in this regard. We have tried to present the material in the clearest possible manner with many helpful examples. *Take advantage of this and read each section of the textbook thoroughly and carefully several times over.* Each re-reading will allow you to see connections among the concepts on a deeper level. *You should read the text with pencil, paper, and a calculator at your side.* Reproduce on your own every computation in every example, so that you truly understand what is presented in the text. *Make notes to yourself as you proceed.* Try as many exercises in each section as possible. There are True/False questions to test your knowledge at the end of each section and in the Review Exercises for Chapters 1 to 7. After pondering these first on your own, compare your answers with the detailed solutions given in the Student Solutions Manual. *Ask your instructor questions about anything that you read that you do not comprehend* — as soon as possible, because each new section continually builds on previous material.

Facility with Proofs: Linear algebra is considered by many instructors as a transitional course from the freshman computationally-oriented calculus sequence to the junior-senior level courses which put much more emphasis on the reading and writing of mathematical proofs. At first it may seem daunting to write your own proofs. However, most of the proofs that you are asked to write for this text are relatively short. Many useful strategies for proof-writing are discussed in Section 1.3. The proofs that are presented in this text are meant to serve as good examples. *Study them carefully.* Remember that each step of a proof must be validated with a proper reason — a theorem that was proven earlier, a definition, or a principle of logic. Pondering carefully over the definitions and theorems in the text is a very valuable use of your time, for only by fully comprehending these can you fully appreciate how to use them in proofs. Learning how to read and write proofs effectively is an important skill that will serve you well in your upper-division mathematics courses and beyond.

Student Solutions Manual: A Student Solutions Manual is available online that contains full solutions for each exercise in the text bearing a ⋆ (those whose answers appear in the back of the textbook). Consequently, this manual contains many useful models for solving various types of problems. The Student Solutions Manual also contains proofs of most of the theorems whose proofs were left to the exercises. These exercises are marked in the text by the symbol ►.

A Light-Hearted Look at Linear Algebra Terms

As students **vector** through the **space** of this text from its **initial point** to its **terminal point**, on a **one-to-one basis**, they will undergo a **real transformation** from the **norm**. An **induction** into the **domain** of linear algebra is **sufficient** to produce a **pivotal** change in their abilities. To **transpose** students with an **empty set** of knowledge into higher **echelons** of understanding, a **nontrivial length** of time is **necessary** — one of the prime **factorizations** to account for in such a **system**.

One **elementary implication** is that the students' success is an **isomorphic reflection** of the **homogeneous** effort they expend on this **complex** material. We can **trace** the **rank** of their achievement to their re**solve** to be a **scalar** of new **distances**. In a **similar** manner, there is a **symmetric result**: their **positive definite** growth is a **function** of their overall **coordinatization** of energy. The **matrix** of thought behind this **parallel** assertion is **proof** that students should avoid the **negative** consequences of **sparse** learning. That is, the **method** of **iterative** study will lead them in an **inverse** way to less **error**, and not **rotate** them into **diagonal tangents** of **zero** worth.

After an **interpolation** of the **kernel** of ideas presented here, the students' **range** of new methods should be **graphically augmented** in a **multiplicity** of ways. We **extrapolate** that one **characteristic** they will attain is a greater **linear independence** in problem-solving. An **associative** feature of this **transition** is that all these new **techniques** should become a **consistent** and **normalized** part of their **identity**.

In **addition**, students will gain a **singular** appreciation of their mathematical skills, so the **resultant skewed** change in their self-**image** should not be of **minor magnitude**, but **complement** them fully. Our **projection** is that the **unique dimensions** of this text will be a **determinant cofactor** in enriching the **span** of their lives, and **translate** them **onto** new **orthogonal** paths of logical truth.

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Symbol Table

$\oplus$	addition on a vector space (unusual)
$\mathcal{A}$	adjoint (classical) of a matrix **A**
I	ampere (unit of current)
$\approx$	approximately equal to
$[\mathbf{A}\|\mathbf{B}]$	augmented matrix formed from matrices **A** and **B**
$\mathbf{p}_L(x)$	characteristic polynomial of a linear operator L
$\mathbf{p}_\mathbf{A}(x)$	characteristic polynomial of a matrix **A**
$\mathcal{A}_{ij}$	cofactor, (i,j), of a matrix **A**
$\overline{z}$	complex conjugate of a complex number z
$\overline{\mathbf{z}}$	complex conjugate of $\mathbf{z} \in \mathbb{C}^n$
$\overline{\mathbf{Z}}$	complex conjugate of $\mathbf{Z} \in \mathcal{M}^{\mathbb{C}}_{mn}$
$\mathbb{C}$	complex numbers, set of
$\mathbb{C}^n$	complex n-vectors, set of (ordered n-tuples of complex numbers)
$g \circ f$	composition of functions f and g
$L_2 \circ L_1$	composition of linear transformations L_1 and L_2
$\mathbf{Z}^*$	conjugate transpose of $\mathbf{Z} \in \mathcal{M}^{\mathbb{C}}_{mn}$
$C^0(\mathbb{R})$	continuous real-valued functions with domain $\mathbb{R}$, set of
$C^1(\mathbb{R})$	continuously differentiable functions with domain $\mathbb{R}$, set of
$[\mathbf{w}]_B$	coordinatization of a vector **w** with respect to a basis B
$\mathbf{x} \times \mathbf{y}$	cross product of vectors **x** and **y**
$f^{(n)}$	derivative, nth, of a function f
$\|\mathbf{A}\|$	determinant of a matrix **A**
δ	determinant of a 2×2 matrix, $ad - bc$
$\mathcal{D}_n$	diagonal $n \times n$ matrices, set of
$\dim(\mathcal{V})$	dimension of a vector space $\mathcal{V}$
$\mathbf{x} \cdot \mathbf{y}$	dot product, or, complex dot product, of vectors **x** and **y**
λ	eigenvalue of a matrix
E_λ	eigenspace corresponding to eigenvalue λ
$\{\ \}$	empty set
a_{ij}	entry, (i,j), of a matrix **A**
$f\colon X \to Y$	function f from a set X (domain) to a set Y (codomain)
$\mathbf{I}, \mathbf{I}_n$	identity matrix; $n \times n$ identity matrix

$\Longleftrightarrow$, iff	if and only if
$f(S)$	image of a set S under a function f
$f(x)$	image of an element x under a function f
i	imaginary number whose square $= -1$
$\Longrightarrow$	implies; if...then
$< \mathbf{x}, \mathbf{y} >$	inner product of $\mathbf{x}$ and $\mathbf{y}$
$\mathbb{Z}$	integers, set of
f^{-1}	inverse of a function f
L^{-1}	inverse of a linear transformation L
$\mathbf{A}^{-1}$	inverse of a matrix $\mathbf{A}$
$\cong$	isomorphic
$\ker(L)$	kernel of a linear transformation L
$\|\|\mathbf{a}\|\|$	length, or norm, of a vector $\mathbf{a}$
$\mathbf{M}_f$	limit matrix of a Markov chain
$\mathbf{p}_f$	limit vector of a Markov chain
$\mathcal{L}_n$	lower triangular $n \times n$ matrices, set of
$\|z\|$	magnitude (absolute value) of a complex number z
$\mathcal{M}_{mn}$	matrices of size $m \times n$, set of
$\mathcal{M}_{mn}^{\mathbb{C}}$	matrices of size $m \times n$ with complex entries, set of
$\mathbf{A}_{BC}$	matrix for a linear transformation w.r.t. ordered bases B, C
$\|\mathbf{A}_{ij}\|$	minor, (i,j), of a matrix $\mathbf{A}$
$\mathbb{N}$	natural numbers, set of
not A	negation of statement A
$\|S\|$	number of elements in a set S
Ω	ohm (unit of resistance)
$(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$	ordered basis containing vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$
$\mathcal{W}^{\perp}$	orthogonal complement of a subspace $\mathcal{W}$
$\perp$	perpendicular to
$\mathcal{P}_n$	polynomials of degree $\leq n$, set of
$\mathcal{P}_n^{\mathbb{C}}$	polynomials of degree $\leq n$ with complex coefficients, set of
$\mathcal{P}$	polynomials, set of all
$\mathbb{R}^+$	positive real numbers, set of
$\mathbf{A}^k$	power, kth, of a matrix $\mathbf{A}$
$f^{-1}(S)$	pre-image of a set S under a function f
$f^{-1}(x)$	pre-image of an element x under a function f
$\mathbf{proj_a b}$	projection of $\mathbf{b}$ onto $\mathbf{a}$
$\mathbf{proj}_{\mathcal{W}}\mathbf{v}$	projection of $\mathbf{v}$ onto a subspace $\mathcal{W}$
$\mathbf{A}^+$	pseudoinverse of a matrix $\mathbf{A}$

range(L)	range of a linear transformation L
rank($\mathbf{A}$)	rank of a matrix $\mathbf{A}$
$\mathbb{R}$	real numbers, set of
$\mathbb{R}^n$	real n-vectors, set of (ordered n-tuples of real numbers)
$\langle i \rangle \leftarrow c \langle i \rangle$	row operation of Type (I)
$\langle i \rangle \leftarrow c \langle j \rangle + \langle i \rangle$	row operation of Type (II)
$\langle i \rangle \leftrightarrow \langle j \rangle$	row operation of Type (III)
$R(\mathbf{A})$	row operation R applied to matrix $\mathbf{A}$
$\odot$	scalar multiplication on a vector space (unusual)
σ_k	singular value, kth, of a matrix
$m \times n$	size of a matrix with m rows and n columns
span(S)	span of a set S
$\mathbf{\Psi}_{ij}$	standard basis vector (matrix) in $\mathcal{M}_{nn}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	standard basis vectors in $\mathbb{R}^3$
$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$	standard basis vectors in $\mathbb{R}^n$; standard basis vectors in $\mathbb{C}^n$
$\mathbf{p}_n$	state vector, nth, of a Markov chain
$\mathbf{A}_{ij}$	submatrix, (i, j), of a matrix $\mathbf{A}$
$\sum$	sum of
trace($\mathbf{A}$)	trace of a matrix $\mathbf{A}$
$\mathbf{A}^T$	transpose of a matrix $\mathbf{A}$
$C^2(\mathbb{R})$	twice continuously differentiable functions with domain $\mathbb{R}$, set of
$\mathcal{U}_n$	upper triangular $n \times n$ matrices, set of
V	volt (unit of voltage)
$\mathbf{O}$; $\mathbf{O}_n$; $\mathbf{O}_{mn}$	zero matrix; $n \times n$ zero matrix; $m \times n$ zero matrix
$\mathbf{0}$; $\mathbf{0}_{\mathcal{V}}$	zero vector in a vector space $\mathcal{V}$

Computational & Numerical Techniques, Applications

The following is a list of the most important computational and numerical techniques and applications of linear algebra presented throughout the text.

Section	Technique / Application
Section 1.1	Resultant Velocity
Section 1.1	Newton's Second Law
Section 1.2	Work (in physics)
Section 1.5	Shipping Cost and Profit
Section 2.1	Gaussian Elimination and Back Substitution
Section 2.1	Curve Fitting
Section 2.2	Gauss-Jordan Row Reduction Method
Section 2.2	Balancing of Chemical Equations
Section 2.3	Determining the Row Space of a Matrix
Section 2.4	Inverse Method (finding the inverse of a matrix)
Section 2.4	Solving a System using the Inverse of the Coefficient Matrix
Section 2.4	Finding the Determinant of a 2×2 Matrix ($ad-bc$ formula)
Section 3.1	Finding the Determinant of a 3×3 Matrix (Basketweaving)
Section 3.1	Finding Areas and Volumes using Determinants
Section 3.2	Determinant of a Matrix by Row Reduction
Section 3.3	Determinant of a Matrix by General Cofactor Expansion
Section 3.3	Cramer's Rule
Section 3.4	Finding Eigenvalues and Eigenvectors for a Matrix
Section 3.4	Diagonalization Method (diagonalizing a square matrix)
Section 4.3	Simplified Span Method (determining span by row reduction)
Section 4.4	Independence Test Method (linear independence by row reduction)
Section 4.6	Enlarging Method (enlarging a linearly independent set to a basis)
Section 4.7	Coordinatization Method (coordinatizing w.r.t. an ordered basis)
Section 4.7	Transition Matrix Method (transition matrix by row reduction)
Section 5.2	Determining the Matrix for a Linear Transformation
Section 5.3	Kernel Method (basis for a kernel of a linear transformation)
Section 5.3	Range Method (basis for the range of a linear transformation)
Section 5.4	Determining if a Linear Transformation is One-to-One or Onto
Section 5.5	Determining if a Linear Transformation is an Isomorphism
Section 5.6	Generalized Diagonalization Method (diagonalizing a linear operator)

Section	Technique / Application
Section 6.1	Gram-Schmidt Process
Section 6.2	Finding the Orthogonal Complement of a Subspace
Section 6.2	Finding the Orthogonal Projection of a Vector onto a Subspace
Section 6.2	Finding the Distance from a Point to a Subspace
Section 6.3	Orthogonal Diagonalization Method
Section 7.2	Gaussian Elimination for Complex Systems
Section 7.2	Gauss-Jordan Row Reduction Method for Complex Systems
Section 7.2	Complex Determinants, Eigenvalues, and Matrix Diagonalization
Section 7.4	Gram-Schmidt Process for Complex Vectors
Section 7.5	Finding the Distance Between Vectors in an Inner Product Space
Section 7.5	Finding the Angle Between Vectors in an Inner Product Space
Section 7.5	Orthogonal Complement of a Subspace in an Inner Product Space
Section 7.5	Orthogonal Projection of a Vector onto an Inner Product Subspace
Section 7.5	Generalized Gram-Schmidt Process (for inner product spaces)
Section 7.5	Fourier Series
Section 8.1	Number of Paths between Vertices in a Graph/Digraph
Section 8.1	Determining if a Graph is Connected
Section 8.2	Finding the Current in a Branch of an Electrical Circuit
Section 8.3	Finding the Least-Squares Polynomial for a Set of Data
Section 8.4	Finding the Steady-State Vector for a Markov Chain
Section 8.5	Encoding/Decoding Messages using Hill Substitution
Section 8.6	Using Rotation of Axes to Graph a Conic Section
Section 8.7	Similarity Method (in computer graphics)
Section 8.8	Solutions of a System of First-Order Linear Differential Equations
Section 8.8	Solutions to Higher-Order Homogeneous Differential Equations
Section 8.9	Least-Squares Solutions for Inconsistent Systems
Section 8.9	Approximate Eigenvalues/Eigenvectors using Inconsistent Systems
Section 8.10	Quadratic Form Method (diagonalizing a quadratic form)
Section 9.1	Partial Pivoting (to avoid roundoff errors when solving systems)
Section 9.1	Jacobi and Gauss-Seidel (Iterative) Methods (for solving systems)
Section 9.2	**LDU** Decomposition Method
Section 9.3	Power Method (dominant eigenvalue of a square matrix)
Section 9.4	**QR** Factorization Method
Section 9.5	Singular Value Decomposition Method
Section 9.5	Calculating the Pseudoinverse of a Matrix
Section 9.5	Digital Imaging (using Singular Value Decomposition)
Appendix D	Decomposing a Matrix as a Product of Elementary Matrices