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Integration and Probability

Springer

Foreword

It is a distinct pleasure to have the opportunity to introduce Professor Malliavin's book to the English-speaking mathematical world.

In recent years there has been a noticeable retreat from the level of abstraction at which graduate-level courses in analysis were previously taught in the United States and elsewhere. In contrast to the practices used in the 1950s and 1960s, when great emphasis was placed on the most general context for integration and operator theory, we have recently witnessed an increased emphasis on detailed discussion of integration over Euclidean space and related problems in probability theory, harmonic analysis, and partial differential equations.

Professor Malliavin is uniquely qualified to introduce the student to analysis with the proper mix of abstract theories and concrete problems. His mathematical career includes many notable contributions to harmonic analysis, complex analysis, and related problems in probability theory and partial differential equations. Rather than developed as a thing-in-itself, the abstract approach serves as a context into which special models can be couched. For example, the general theory of integration is developed at an abstract level, and only then specialized to discuss the Lebesgue measure and integral on the real line. Another important area is the entire theory of probability, where we prefer to have the abstract model in mind, with no other specialization than total unit mass. Generally, we learn to work at an abstract level so that we can specialize when appropriate.

A cursory examination of the contents reveals that this book covers most of the topics that are familiar in the first graduate course on analysis. It also treats topics that are not available elsewhere in textbook form. A notable example is Chapter V, which deals with Malliavin's stochastic calculus of variations developed in the context of Gaussian measure spaces. Originally inspired by the desire to obtain a probabilistic proof of Hörmander's theorem on the smoothness of the solutions of second-order hypoelliptic differential equations, the subject has found a life of its own. This is partly due to Malliavin and his followers' development of a suitable notion of "differentiable function" on a Gaussian measure space. The novice should be warned that this notion of differentiability is not easily related to the more conventional notion of differentiability in courses on manifolds. Here we have a family of Sobolev spaces of "differentiable functions" over the measure space, where the definition is global, in terms of the Sobolev norms. The finite-dimensional Sobolev spaces are introduced through translation operators, and immediately generalizes to the infinite-dimensional case. The main theorem of the subject states that if a differentiable vector-valued function has enough "variation", then it induces a smooth measure on Euclidean space.

Such relations illustrate the interplay between the "upstairs" and the "downstairs" of analysis. We find the natural proof of a theorem in real analysis (smoothness of a measure) by going up to the infinite-dimensional Gaussian measure space where the measure is naturally defined. This interplay of ideas can also be found in more traditional forms of finite-dimensional real analysis, where we can better understand and prove formulas and theorems on special functions on the real line by going up to the higher-dimensional geometric problems from which they came by "projection"; Bessel and Legendre functions provide some elementary examples of such phenomena.

The mathematical public owes an enormous debt of gratitude to Leslie Kay, whose superlative efforts in editing and translating this text have been accomplished with great speed and accuracy.

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Preface

We plan to survey various extensions of Lebesgue theory in contemporary analysis: the abstract integral, Radon measures, Fourier analysis, Hilbert spectral analysis, Sobolev spaces, pseudo-differential operators, probability, martingales, the theory of differentiation, and stochastic calculus of variations.

In order to give complete proofs within the limits of this book, we have chosen an axiomatic method of exposition; the interest of the concepts introduced will become clear only after the reader has encountered examples later in the text. For instance, the first chapter deals with the abstract integral, but the reader does not see a nontrivial example of the abstract theory until the Lebesgue integral is introduced in Chapter II. This axiomatic approach is now familiar in topology; it should not cause difficulties in the theory of integration.

In addition, we have tried as much as possible to base each theory on the results of the theories presented earlier. This structure permits an economy of means, furnishes interesting examples of applications of general theorems, and above all illustrates the unity of the subject. For example, the Radon-Nikodym theorem, which could have appeared at the end of Chapter I, is treated at the end of Chapter IV as an example of the theory of martingales; we then obtain the stronger result of convergence almost everywhere. Similarly, conditional probabilities are treated using (i) the theory of Radon measures and (ii) a general isomorphism theorem showing that there exists only one model of a nonatomic separable measure space, namely $\mathbf{R}$ equipped with Lebesgue measure. Furthermore, the spectral theory of unitary operators on an abstract Hilbert space is derived from

Bochner's theorem characterizing Fourier series of measures. The treatment in Chapter V of Sobolev spaces over a probability space parallels that in Chapter III of Sobolev spaces over $\mathbf{R}^n$.

In the detailed table of contents, the reader can see how the book is organized. It is easy to read only selected parts of the book, depending on the results one hopes to reach; at the beginning of the book, as a reader's guide, there is a diagram showing the interdependence of the different sections. There is also an index of terms at the end of the work. Certain parts of the text, which can be skipped on a first reading, are printed in smaller type.

Readers interested in probability theory can focus essentially on Chapters I, IV, and V; those interested in Fourier analysis, essentially on Chapters I and III. Chapter III can be read in different ways, depending on whether one is interested in partial differential equations or in spectral analysis.

The book includes a variety of exercises by Gérard Letac. Detailed solutions can be found in *Exercises and Solutions Manual for Integration and Probability* by Gérard Letac, Springer-Verlag, 1995. The upcoming book *Stochastic Analysis* by Paul Malliavin, Grundlehren der Mathematischen Wissenschaften, volume 313, Springer-Verlag, 1995, is meant for second-year graduate students who are planning to continue their studies in probability theory.

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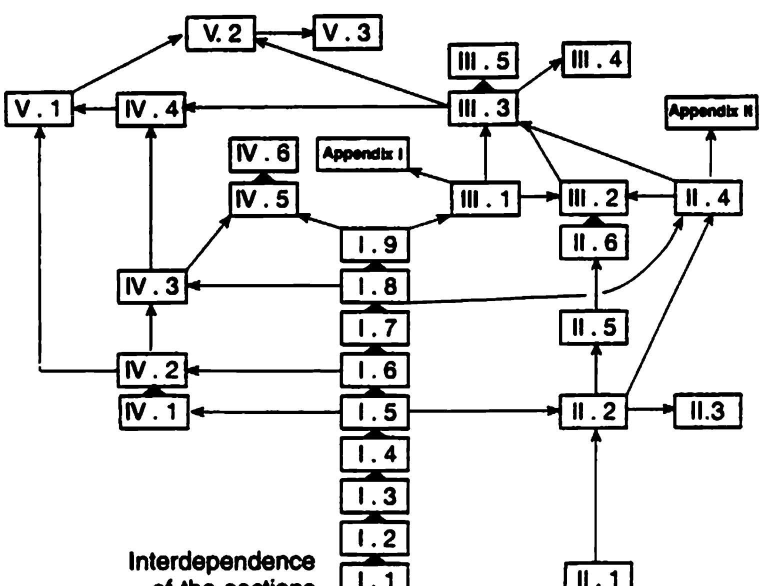

Interdependence of the sections

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