An Undergraduate Introduction to Financial Mathematics

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Preface

This book is intended for an audience with an undergraduate level of exposure to calculus through elementary multivariable calculus. The book assumes no background on the part of the reader in probability or statistics. One of my objectives in writing this book was to create a readable, reasonably self-contained introduction to financial mathematics for people wanting to learn some of the basics of option pricing and hedging. My desire to write such a book grew out of the need to find an accessible book for undergraduate mathematics majors on the topic of financial mathematics. I have taught such a course now three times and this book grew out of my lecture notes and reading for the course. New titles in financial mathematics appear constantly, so in the time it took me to compose this book there may have appeared several superior works on the subject. Knowing the amount of work required to produce this book, I stand in awe of authors such as those.

This book consists of ten chapters which are intended to be read in order, though the well-prepared reader may be able to skip the first several with no loss of understanding in what comes later. The first chapter is on interest and its role in finance. Both discretely compounded and continuously compounded interest are treated there. The book begins with the theory of interest because this topic is unlikely to scare off any reader no matter how long it has been since they have done any formal mathematics.

The second and third chapters provide an introduction to the concepts of probability and statistics which will be used throughout the remainder of the book. Chapter Two deals with discrete random variables and emphasizes the use of the binomial random variable. Chapter Three introduces continuous random variables and emphasizes the similarities and differences between discrete and continuous random variables. The normal random variable and the close related lognormal random variable are introduced and explored in the latter chapter.

In the fourth chapter the concept of arbitrage is introduced. For readers already well versed in calculus, probability, and statistics, this is the first material which may be unfamiliar to them. The assumption that financial calculations are carried out in an "arbitrage free" setting pervades the remainder of the book. The lack of arbitrage opportunities in financial transactions ensures that it is not possible to make a risk free profit. This chapter includes a discussion of the result from linear algebra and operations research known as the Duality Theorem of Linear Programming.

The fifth chapter introduces the reader to the concepts of random walks and Brownian motion. The random walk underlies the mathematical model of the value of securities such as stocks and other financial instruments whose values are derived from securities. The choice of material to present and the method of presentation is difficult in this chapter due to the complexities and subtleties of stochastic processes. I have attempted to introduce stochastic processes in an intuitive manner and by connecting elementary stochastic models of some processes to their corresponding deterministic counterparts. Itô's Lemma is introduced and an elementary proof of this result is given based on the multivariable form of Taylor's Theorem. Readers whose interest is piqued by material in Chapter Five should consult the bibliography for references to more comprehensive and detailed discussions of stochastic calculus.

Chapter Six introduces the topic of options. Both European and American style options are discussed though the emphasis is on European options. Properties of options such as the Put/Call Parity formula are presented and justified. In this chapter we also derive the partial differential equation and boundary conditions used to price European call and put options. This derivation makes use of the earlier material on arbitrage, stochastic processes and the Put/Call Parity formula.

The seventh chapter develops the solution to the Black-Scholes PDE. There are several different methods commonly used to derive the solution to the PDE and students benefit from different aspects of each derivation. The method I choose to solve the PDE involves the use of the Fourier Transform. Thus this chapter begins with a brief discussion of the Fourier and Inverse Fourier Transforms and their properties. Most three- or four-semester elementary calculus courses include at least an optional section on the Fourier Transform, thus students will have the calculus background necessary to follow this discussion. It also provides exposure to the Fourier

Transform for students who will be later taking a course in PDEs and more importantly exposure for students who will not take such a course. After completing this derivation of the Black-Scholes option pricing formula students should also seek out other derivations in the literature for the purposes of comparison.

Chapter Eight introduces some of the commonly discussed partial derivatives of the Black-Scholes option pricing formula. These partial derivatives help the reader to understand the sensitivity of option prices to movements in the underlying security's value, the risk-free interest rate, and the volatility of the underlying security's value. The collection of partial derivatives introduced in this chapter is commonly referred to as "the Greeks" by many financial practitioners. The Greeks are used in the ninth chapter on hedging strategies for portfolios. Hedging strategies are used to protect the value of a portfolio against movements in the underlying security's value, the risk-free interest rate, and the volatility of the underlying security's value. Mathematically the hedging strategies remove some of the low order terms from the Black-Scholes option pricing formula making it less sensitive to changes in the variables upon which it depends. Chapter Nine will discuss and illustrate several examples of hedging strategies.

Chapter Ten extends the ideas introduced in Chapter Nine by modeling the effects of correlated movements in the values of investments. The tenth chapter discusses several different notions of optimality in selecting portfolios of investments. Some of the classical models of portfolio selection are introduced in this chapter including the Capital Assets Pricing Model (CAPM) and the Minimum Variance Portfolio.

It is the author's hope that students will find this book a useful introduction to financial mathematics and a springboard to further study in this area. Writing this book has been hard, but intellectually rewarding work.

During the summer of 2005 a draft version of this manuscript was used by the author to teach a course in financial mathematics. The author is indebted to the students of that class for finding numerous typographical errors in that earlier version which were corrected before the camera ready copy was sent to the publisher. The author wishes to thank Jill Bachstadt, Jason Buck, Mark Elicker, Kelly Flynn, Jennifer Gomulka, Nicole Hundley, Alicia Kasif, Stephen Kluth, Patrick McDevitt, Jessica Paxton, Christopher Rachor, Timothy Refi, Pamela Wentz, Joshua Wise, and Michael Zrncic.

A list of errata and other information related to this book can be found at a web site I created:

http://banach.millersville.edu/~bob/book/

Please feel free to share your comments, criticism, and (I hope) praise for this work through the email address that can be found at that site.

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Lancaster, PA, USA
October 31, 2005

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