Real Analysis

(Fourth Edition)

H. L. Royden

P. M. Fitzpatrick

China Machine Press

Preface

The first three editions of H.L.Royden's *Real Analysis* have contributed to the education of generations of mathematical analysis students. This fourth edition of *Real Analysis* preserves the goal and general structure of its venerable predecessors—to present the measure theory, integration theory, and functional analysis that a modern analyst needs to know.

The book is divided the three parts: Part I treats Lebesgue measure and Lebesgue integration for functions of a single real variable; Part II treats abstract spaces—topological spaces, metric spaces, Banach spaces, and Hilbert spaces; Part III treats integration over general measure spaces, together with the enrichments possessed by the general theory in the presence of topological, algebraic, or dynamical structure.

The material in Parts II and III does not formally depend on Part I. However, a careful treatment of Part I provides the student with the opportunity to encounter new concepts in a familiar setting, which provides a foundation and motivation for the more abstract concepts developed in the second and third parts. Moreover, the Banach spaces created in Part I, the L^p spaces, are one of the most important classes of Banach spaces. The principal reason for establishing the completeness of the L^p spaces and the characterization of their dual spaces is to be able to apply the standard tools of functional analysis in the study of functionals and operators on these spaces. The creation of these tools is the goal of Part II.

NEW TO THE EDITION

- This edition contains 50% more exercises than the previous edition
- Fundamental results, including Egoroff's Theorem and Urysohn's Lemma are now proven in the text.
- The Borel-Cantelli Lemma, Chebychev's Inequality, rapidly Cauchy sequences, and the continuity properties possessed both by measure and the integral are now formally presented in the text along with several other concepts.

There are several changes to each part of the book that are also noteworthy:

Part I

- The concept of uniform integrability and the Vitali Convergence Theorem are now presented and make the centerpiece of the proof of the fundamental theorem of integral calculus for the Lebesgue integral
- A precise analysis of the properties of rapidly Cauchy sequences in the $L^p(E)$ spaces, $1 \leq p \leq \infty$, is now the basis of the proof of the completeness of these spaces
- Weak sequential compactness in the $L^p(E)$ spaces, $1 \leq p \leq \infty$, is now examined in detail and used to prove the existence of minimizers for continuous convex functionals.

Part II

- General structural properties of metric and topological spaces are now separated into two brief chapters in which the principal theorems are proven.
- In the treatment of Banach spaces, beyond the basic results on bounded linear operators, compactness for weak topologies induced by the duality between a Banach space and its dual is now examined in detail.
- There is a new chapter on operators in Hilbert spaces, in which weak sequential compactness is the basis of the proofs of the Hilbert-Schmidt theorem on the eigenvectors of a compact symmetric operator and the characterization by Riesz and Schuader of linear Fredholm operators of index zero acting in a Hilbert space.

Part III

- General measure theory and general integration theory are developed, including the completeness, and the representation of the dual spaces, of the $L^p(X, \mu)$ spaces for, $1 \leq p \leq \infty$. Weak sequential compactness is explored in these spaces, including the proof of the Dunford-Pettis theorem that characterizes weak sequential compactness in $L^1(X, \mu)$.
- The relationship between topology and measure is examined in order to characterize the dual of $C(X)$, for a compact Hausdorff space X. This leads, via compactness arguments, to (i) a proof of von Neumann's theorem on the existence of unique invariant measures on a compact group and (ii) a proof of the existence, for a mapping on a compact Hausdorf space, of a probability measure with respect to which the mapping is ergodic.

The general theory of measure and integration was born in the early twentieth century. It is now an indispensable ingredient in remarkably diverse areas of mathematics, including probability theory, partial differential equations, functional analysis, harmonic analysis, and dynamical systems. Indeed, it has become a unifying concept. Many different topics can agreeably accompany a treatment of this theory. The companionship between integration and functional analysis and, in particular, between integration and weak convergence, has been fostered here: this is important, for instance, in the analysis of nonlinear partial differential equations (see L.C. Evans' book *Weak Convergence Methods for Nonlinear Partial Differential Equations* [AMS, 1998]).

The bibliography lists a number of books that are not specifically referenced but should be consulted for supplementary material and different viewpoints. In particular, two books on the interesting history of mathematical analysis are listed.

SUGGESTIONS FOR COURSES: FIRST SEMESTER

In Chapter 1, all the background elementary analysis and topology of the real line needed for Part I is established. This initial chapter is meant to be a handy reference. Core material comprises Chapters 2, 3, and 4, the first five sections of Chapter 6, Chapter 7, and the first section of Chapter 8. Following this, selections can be made: Sections 8.2–8.4 are interesting for students who will continue to study duality and compactness for normed linear spaces, while Section 5.3 contains two jewels of classical analysis, the characterization of Lebesgue integrability and of Riemann integrability for bounded functions.

SUGGESTIONS FOR COURSES: SECOND SEMESTER

This course should be based on Part III. Initial core material comprises Section 17.1, Section 18.1–18.4, and Sections 19.1–19.3. The remaining sections in Chapter 17 may be covered at the beginning or as they are needed later: Sections 17.3–17.5 before Chapter 20, and Section 17.2 before Chapter 21. Chapter 20 can then be covered. None of this material depends on Part II. Then several selected topics can be chosen, dipping into Part II as needed.

- Suggestion 1: Prove the Baire Category Theorem and its corollary regarding the partial continuity of the pointwise limit of a sequence of continuous functions (Theorem 7 of Chapter 10), infer from the Riesz-Fischer Theorem that the Nikodym metric space is complete (Theorem 23 of Chapter 18), prove the Vitali-Hahn-Saks Theorem and then prove the Dunford-Pettis Theorem.
- Suggestion 2: Cover Chapter 21 (omitting Section 20.5) on Measure and Topology, with the option of assuming the topological spaces are metrizable, so 20.1 can be skipped.
- Suggestion 3: Prove Riesz's Theorem regarding the closed unit ball of an infinite dimensional normed linear space being noncompact with respect to the topology induced by the norm. Use this as a motivation for regaining sequential compactness with respect to weaker topologies, then use Helley's Theorem to obtain weak sequential compactness properties of the $L^p(X, \mu)$ spaces, $1 < p < \infty$, if $L^q(X, \mu)$ is separable and, if Chapter 21 has already been covered, weak-$*$ sequential compactness results for Radon measures on the Borel σ-algebra of a compact metric space.

SUGGESTIONS FOR COURSES: THIRD SEMESTER

I have used Part II, with some supplemental material, for a course on functional analysis, for students who had taken the first two semesters; the material is tailored, of course, to that chosen for the second semester. Chapter 16 on bounded linear operators on a Hilbert space may be covered right after Chapter 13 on bounded linear operators on a Banach space, since the results regarding weak sequential compactness are obtained directly from the existence of an orthogonal complement for each closed subspace of a Hilbert space. Part II should be interlaced with selections from Part III to provide applications of the abstract space theory to integration. For instance, reflexivity and weak compactness can be considered in general $L^p(X, \mu)$ spaces, using material from Chapter 19. The above suggestion 1 for the second semester course can be taken in the third semester rather than the second, providing a truly striking application of the Baire Category Theorem. The establishment, in Chapter 21, of the representation of the dual of $C(X)$, where X is a compact Hausdorff space, provides another collection of spaces, spaces of signed Radon measures, to which the theorems of Helley, Alaoglu, and Krein-Milman apply. By covering Chapter 22 on Invariant Measures, the student will encounter applications of Alaoglu's Theorem and the Krein-Milman Theorem to prove the existence of Haar measure on a compact group and the existence of measures with respect to which a mapping is ergodic (Theorem 14 of Chapter 22), and an application of Helley's Theorem to establish the existence of invariant measures (the Bogoliubov-Krilov Theorem).

I welcome comments at pmf@math.umd.edu. A list of errata and remarks will be placed on www.math.umd.edu/~pmf/RealAnalysis.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge my indebtedness to teachers, colleagues, and students. A penultimate draft of the entire manuscript was read by Diogo Arsénio, whom I warmly thank for his observations and suggestions, which materially improved that draft. Here in my mathematical home, the University of Maryland, I have written notes for various analysis courses, which have been incorporated into the present edition. A number of students in my graduate analysis classes worked through parts of drafts of this edition, and their comments and suggestions have been most helpful: I thank Avner Halevy, Kevin McGoff, and Himanshu Tiagi. I am pleased to acknowledge a particular debt to Brendan Berg who created the index, proofread the final manuscript, and kindly refined my tex skills. I have benefited from conversations with many friends and colleagues; in particular, with Diogo Arsénio, Michael Boyle, Michael Brin, Craig Evans, Manos Grillakis, Brian Hunt, Jacobo Pejsachowicz, Eric Slud, Robert Warner, and Jim Yorke. Publisher and reviewers: J. Thomas Beale, Duke University; Richard Carmichael, Wake Forest University; Michael Goldberg, Johns Hopkins University; Paul Joyce, University of Idaho; Dmitry Kaliuzhnyi-Verbovetskyi, Drexel University; Giovanni Leoni, Carnegie Mellon University; Bruce Mericle, Mankato State University; Stephen Robinson, Wake Forest University; Martin Schechter, University of California-Irvine; James Stephen White, Jacksonville State University; and Shanshuang Yang, Emory University.

Patrick M. Fitzpatrick
College Park, MD
November, 2009

Contents