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Essay on Macroeconomics and Expectations

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by

Giovanni Nicolò

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ABSTRACT OF THE DISSERTATION

Essay on Macroeconomics and Expectations

by

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My dissertation focuses on the interactions between the conduct of U.S. monetary policy and the expectations formed by households, firms and public institutions about the state of economy. The first two chapters develop new methods that I use in the subsequent chapters to study how expectations formed by economic agents about future economic conditions affect a given economy. The second chapter considers and extends the work in Farmer (2012a) to explain U.S. post-war data, and shows that it outperforms conventional economic theories due to its ability to account for persistent movements in the data. The last chapter explores how the effectiveness of monetary policy changed in the U.S. post-war period, and I provide evidence that since the early 1980's the monetary authority implemented policies that reduced economic uncertainty deriving from unforeseen changes in the expectations about future inflation.

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To my parents,

whose love, support and sacrifices

taught me the value of family.

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Introduction

Expectations and monetary policy are closely related. While the implementation of monetary policies crucially affects the expectations formed by the private sector, its effectiveness heavily depends on the beliefs held by the latter. My dissertation deepens the understanding of these interactions by providing evidence in four chapters. The first two chapters develop new methods applied in the subsequent chapters, and that allow to solve and estimate models in which expectations could play a relevant role. The third chapter considers the paradigm developed in Farmer (2012a) to show that the high persistence in the inflation rate, the nominal interest rate and the output gap in the United States is explained by an economy in which expectations about the future economic conditions have a potentially permanent effect on current economic outcomes. Finally, in the last chapter , I study the conduct of U.S. monetary policy during the post-war period, and show that since the early 1980's the monetary authority implemented policies which suppressed the possibility for expectations to generate economic uncertainty.

Chapter I & II: Methods to Deal with Expectations in Macroeconomic Models
Chapter I and II present methodologies implemented in Chapter III and IV. Chapter I draws from the paper **"Solving and Estimating Indeterminate DSGE Models,"** published in the *Journal of Economic Dynamics and Control*, and coauthored with Roger E. A. Farmer and Vadim Khramov, (Farmer et al., 2015). The second chapter builds on the previous paper and constitutes new research developed in the joint paper with Francesco Bianchi, **"A Generalized Approach to Indeterminacy in Linear Rational Expectations Models,"** (Bianchi and Nicolò, 2017).

The first chapter proposes a method for solving and estimating linear rational expectations models that exhibit indeterminacy and we provide step-by-step guidelines for implementing this method in the Matlab-based packages Dynare and Gensys. Our method redefines a subset of expectational errors as new fundamentals. This redefinition allows us to treat indeterminate models as determinate and to apply standard solution algorithms. We prove that our method is equivalent to the solution method proposed by Lubik and Schorfheide (2003, 2004), and using the New-Keynesian model described in Lubik and Schorfheide (2004), we demonstrate how to apply our theoretical results with a practical exercise.

The second chapter then proposes a generalized approach to deal with the problem of indeterminacy in Linear Rational Expectations models. Our method consists of augmenting the original model with a set of auxiliary exogenous equations that are used to provide the adequate number of explosive roots in presence of indeterminacy. Using our approach, the researcher can estimate the model by using standard packages without restricting the estimates to a certain area of the parameter space, and she can test more easily whether the data favor a model specification in which expectations play a fundamental role in the economy.

Chapter III: Monetary Policy, Expectations and the U.S. Economy in the Long-Run

The Great Recession revived a familiar debate among academics and policy makers. The crisis had long-lasting, if not permanent, effects on the U.S. economy, and this fact is consistent with the evidence provided in many econometric studies. U.S. macroeconomic data are well-described by non-stationary time series: after a given economy suffers a crisis, it experiences permanent losses in terms of GDP, consumption and investment, while the unemployment rate, the inflation rate and the interest rate adjust to a new level. Conventional theories cannot easily account for these facts because they assume that key macroeconomic variables converge to their long-run trends. There is no possibility of observing permanent effects of transitory shocks.

The third chapter draws from the paper coauthored with Professor Roger E. A. Farmer, titled **"Keynesian Economics Without the Phillips Curve,"** (Farmer and Nicolò, 2018). We extend Farmer's (2012a) Monetary (FM) model to explain the observed persistence in the inflation rate, the nominal interest rate and the output gap in the United States. The model is a three-equation NK model in which the Phillips curve is replaced by a belief function. This is an equation in which expectations about the future growth rate of nominal GDP are determined by observations of current nominal GDP. In the FM model, structural shocks have permanent effects on the economy. Central to this result is the idea that the expectations of households, firms and government are fundamentals that determine the current state of the economy.

In the conventional NK model, policies that alter aggregate demand have no impact on real economic activity in the long-run. In the FM model, policies that target aggregate demand have permanent, long-lasting effects on output and unemployment. Government interventions that increase aggregate demand are potentially powerful tool to increase employment not just temporarily, but in the long-run. Using Bayesian techniques, we show that the FM model fits the U.S. post-war data better the conventional NK theory, and we argue that the improved empirical performance stems from its ability to account for persistent movements in the data.

In future works, I plan to incorporate the innovation of the FM model into a medium-scale model to ask if the cointegrating feature of the FM model can better explain the data in the post-war period relative to a conventional NK model that displays self-stabilizing properties around a unique long-run equilibrium.

Chapter IV: Monetary Policy, Expectations and Business Cycles in the U.S. Post-War Period

In the fourth and last chapter, I study the interactions between monetary policy and expectations in the United States during the post-war period. From the late 1950s through the 1970s, the U.S. economy experienced high volatility, and inflation was high and rising. Since the early 1980s, the macroeconomy was less volatile, and inflation was low and stable. The conduct of monetary policy before 1980 is at odds with the objective of the Federal Reserve to achieve a low and stable inflation rate and stabilize unemployment and growth.

I estimate a conventional medium-scale New-Keynesian (NK) model using Bayesian techniques. In line with previous studies based on small-scale models, I find that the change in the behavior of the data is associated with a change in the conduct of monetary policy. Prior to 1980, monetary policy in the United States was not aggressive enough to stabilize inflation, output and employment. Since the early 1980s, the monetary authority acted more systematically to achieve a low and stable inflation rate.

The main contribution of the paper is to conduct a quantitative analysis of the reasons for which a monetary policy that fails to stabilize the inflation and output growth rationalizes the empirical properties of the data before 1980. In response to structural shocks, inflation expectations are persistently de-anchored from the long-run inflation target of the monetary authority. This mechanism generates an additional source of persistence that explains the run-up in inflation since the early 1960s until the late 1970s. In particular, I find that positive productivity shocks during the 1960s induced persistent inflationary pressures via the formation of self-fulfilling inflationary expectations. The oil crisis that hit the U.S. economy in the mid-1970s also triggered mark-up shocks that prompted additional, sudden inflationary episodes.

Relative to the previous literature, I show that non-fundamental disturbances have no quantitative role in explaining the high macroeconomic volatility before 1980. When the monetary authority fails to stabilize inflation and the macroeconomy, unexpected changes in expectations constitute an additional, non-fundamental source of uncertainty. In my quantitative analysis, I find that the high volatility of the macroeconomy observed prior to 1980 is explained exclusively by structural disturbances and non-fundamental shocks have no quantitative relevance.

Part I

Solving and Estimating Indeterminate DSGE Models

It is well known that linear rational expectations (LRE) models can have an indeterminate set of equilibria under realistic parameter choices. Lubik and Schorfheide (2003) provided an algorithm that computes the complete set of indeterminate equilibrium, but their approach has not yet been implemented in standard software packages and has not been widely applied in practice. In this paper, we propose an alternative methodology based on the idea that a model with an indeterminate set of equilibria is an incomplete model. We propose to close a model of this kind by treating a subset of the non-fundamental errors as newly defined fundamentals.

Our method builds on the approach of Sims (2001b) who provided a widely used computer code, Gensys, implemented in Matlab, to solve for the reduced form of a general class of linear rational expectations (LRE) models. Sims's code classifies models into three groups; those with a unique rational expectations equilibrium, those with an indeterminate set of rational expectations equilibria, and those for which no bounded rational expectations equilibrium exists. By moving non-fundamental errors to the set of fundamental shocks, we select a unique equilibrium, thus allowing the modeler to apply standard solution algorithms. We provide step-by-step guidelines for implementing our method in the Matlab-based software programs Dynare (Adjemian et al., 2011) and Gensys (Sims, 2001b).

Our paper is organized as follows. In Section 1, we provide a brief literature survey and in Section 2 we review solution methods for indeterminate models. In Section 3, we discuss the choice of which expectational errors to redefine as fundamental and we prove that all possible alternative selections have the same likelihood. Section 4 compares our method to the work of Lubik and Schorfheide (2003) and establishes an equivalence result between the two approaches. In Section 5, we apply our method to the New-Keynesian model described in Lubik and Schorfheide (2004) and we show how to apply our method using Gensys to simulated data. Section 6 provides step-by-step guidelines for implementing our method in the popular software package, Dynare,[1] and Section 7 provides a brief conclusion.

1 Related Literature

Blanchard and Kahn (1980a) showed that a LRE model can be written as a linear combination of backward-looking and forward-looking solutions. Since then, a number of alternative approaches for solving linear rational expectations models have emerged (King and Watson, 1998; Klein, 2000; Uhlig, 1999; Sims, 2001b). These methods provide a solution if the equilibrium is unique, but there is considerable confusion about how to handle the indeterminate case. Some methods fail in the case of a non-unique solution, for example, Klein (2000), while others, e.g. Sims (2001b), generate one solution with a warning message.

All of these solution algorithms are based on the idea that, when there is a unique determinate rational expectations equilibrium, the model's forecast errors are uniquely defined by the fundamental shocks. These errors must be chosen in a way that eliminates potentially explosive dynamics of the state variables of the model.

McCallum (1983) has argued that a model with an indeterminate set of equilibria is incompletely specified and he recommends a procedure, the minimal state variable solution, for selecting one of the many possible equilibria in the indeterminate case. Farmer (1999) has argued instead, that we should exploit the properties of indeterminate models to help understand data. Farmer and Guo (1995) took up that challenge by studying a model where indeterminacy arises from a technology with increasing returns-to-scale, and Lubik and Schorfheide (2004), developed methods for distinguishing determinate from indeterminate

[1]Dynare is a Matlab-based software platform for handling a wide class of economic models, in particular dynamic stochastic general equilibrium (DSGE). Visit www.dynare.org for details.

models which they applied to a New-Keynesian monetary model. There is a growing body of literature, see, for example, Belaygorod and Dueker (2009); Fanelli (2012); Castelnuovo and Fanelli (2014); Hirose (2011); Zheng and Guo (2013); Bilbiie and Straub (2013), that directly tackles the econometric challenges posed by indeterminacy. This literature offers the possibility for the theoretical work, surveyed in Benhabib and Farmer (1999), to be directly compared with conventional classical and new-Keynesian approaches in which equilibria are assumed to be locally unique.

The empirical importance of indeterminacy began with the work of Benhabib and Farmer (1994) who established that a standard one-sector growth model with increasing returns displays an indeterminate steady state and Farmer and Guo (1994) who exploited that property to generate business cycle models driven by self-fulfilling beliefs. More recent New-Keynesian models have been shown to exhibit indeterminacy if the monetary authority does not increase the nominal interest rate enough in response to higher inflation (see, for example, Clarida et al. (2000a); Kerr and King (1996)). Our estimation method should be of interest to researchers in both literatures.

2 Solving Linear Rational Expectations Models

Consider the following k-equation LRE model. We assume that $X_t \in R^k$ is a vector of deviations from means of some underlying economic variables. These may include predetermined state variables, for example, the stock of capital, non-predetermined control variables, for example, consumption; and expectations at date t of both types of variables.

We assume that z_t is an $l \times 1$ vector of exogenous, mean-zero shocks and η_t is a $p \times 1$ vector of endogenous shocks.[2] The matrices Γ_0 and Γ_1 are of dimension $k \times k$, possibly singular, Ψ and Π are respectively, $k \times l$ and $k \times\ p$ known matrices.

[2]Sims (2001b) allows z_t to be autoregressive with non zero conditional expectation. We assume, instead, that z_t always has zero conditional mean. That assumption is unrestrictive since an autoregressive error can be written in our form by defining a new state variable, $\tilde{z}_t$ and letting the innovation of the original variable, z_t, be the new fundamental shock.

Using the above definitions, we will study the class of linear rational expectations models described by Equation (2.1),

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi z_t + \Pi \eta_t. \tag{2.1}$$

Sims (2001b) shows that this way of representing a LRE is very general and most LRE models that are studied in practice by economists can be written in this form. We assume that

$$E_{t-1}\left(z_t\right) = 0, \quad \text{and} \quad E_{t-1}\left(\eta_t\right) = 0. \tag{2.2}$$

We define the $l \times l$ matrix Ω_{zz},

$$E_{t-1}\left(z_t z_t^T\right) = \Omega_{zz}, \tag{2.3}$$

which represents the covariance matrix of the exogenous shocks. We refer to these shocks as predetermined errors, or equivalently, predetermined shocks. The second set of shocks, η_t, has dimension p. Unlike the z_t, these shocks are endogenous and are determined by the solution algorithm in a way that eliminates the influence of the unstable roots of the system. In many important examples, the $\eta_{i,t}$ have the interpretation of expectational errors and, in those examples,

$$\eta_{i,t} = X_{i,t} - E_{t-1}\left(X_{i,t}\right). \tag{2.4}$$

2.1 The QZ Decomposition

Sims (2001b) shows how to write equation (2.1) in the form

$$\overbrace{\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}}^{S} \overbrace{\begin{bmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} \end{bmatrix}}^{\tilde{X}_t} = \overbrace{\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}}^{T} \overbrace{\begin{bmatrix} \tilde{X}_{1,t-1} \\ \tilde{X}_{2,t-1} \end{bmatrix}}^{\tilde{X}_{t-1}} + \overbrace{\begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix}}^{\tilde{\Psi}} z_t + \overbrace{\begin{bmatrix} \tilde{\Pi}_1 \\ \tilde{\Pi}_2 \end{bmatrix}}^{\tilde{\Pi}} \eta_t \tag{2.5}$$

where the matrices S, T, $\tilde{\Psi}$ and $\tilde{\Pi}$ and the transformed variables $\tilde{X}_t$ are defined as follows. Let

$$\Gamma_0 = QSZ^T, \quad \text{and} \quad \Gamma_1 = QTZ^T, \tag{2.6}$$

be the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$ where Q and Z are $k \times k$ orthonormal matrices and S and T are upper triangular and possibly complex.

The QZ decomposition is not unique. The diagonal elements of S and T are called the *generalized eigenvalues* of $\{\Gamma_0, \Gamma_1\}$ and Sims's algorithm chooses one specific decomposition that orders the equations so that the absolute values of the ratios of the generalized eigenvalues are placed in increasing order that is,

$$|t_{jj}| \,/\, |s_{jj}| \geq |t_{ii}| \,/\, |s_{ii}| \;\text{ for } j > i. \tag{2.7}$$

Sims proceeds by partitioning S, T, Q and Z as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \tag{2.8}$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \tag{2.9}$$

where the first block contains all the equations for which $|t_{jj}| / |s_{jj}| < 1$ and the second block, all those for which $|t_{jj}| / |s_{jj}| \geq 1$. The transformed variables $\tilde{X}_t$ are defined as

$$\tilde{X}_t = Z^T X_t, \tag{2.10}$$

and the transformed parameters as

$$\tilde{\Psi} = Q^T \Psi, \quad \text{and} \quad \tilde{\Pi} = Q^T \Pi. \tag{2.11}$$

2.2 Using the QZ decomposition to solve the model

The model is said to be determinate if Equation (2.5) has a unique bounded solution. To establish existence of at least one bounded solution we must eliminate the influence of all of the unstable roots; by construction, these are contained in the second block,

$$\tilde{X}_{2,t} = S_{22}^{-1} T_{22} \tilde{X}_{2,t-1} + S_{22}^{-1} \left(\tilde{\Psi}_2 z_t + \tilde{\Pi}_2 \eta_t \right), \tag{2.12}$$

since the eigenvalues of $S_{22}^{-1} T_{22}$ are all greater than or equal to one in absolute value. Hence a bounded solution, if it exists, will set

$$\tilde{X}_{2,0} = 0, \tag{2.13}$$

and

$$\tilde{\Psi}_2 z_t + \tilde{\Pi}_2 \eta_t = 0. \tag{2.14}$$

Since the elements of $\tilde{X}_{2,t}$ are linear combinations of $X_{2,t}$, a necessary condition for the existence of a solution to equation (2.14) is that there are at least as many non-predetermined variables as unstable generalized eigenvalues. A sufficient condition is that the columns of $\tilde{\Pi}_2$ in the matrix,

$$\left[\begin{array}{cc} \tilde{\Psi}_2 & \tilde{\Pi}_2 \end{array} \right], \tag{2.15}$$

are linearly independent so that there is at least one solution to Equation (2.14) for the endogenous shocks, η_t, as a function of the fundamental shocks, z_t. In the case that $\tilde{\Pi}_2$ is square and non-singular, we can write the solution for η_t as

$$\eta_t = -\tilde{\Pi}_2^{-1}\tilde{\Psi}_2 z_t. \tag{2.16}$$

More generally, Sims' code checks for existence using the singular value decomposition of (2.15).

To find a solution for $\tilde{X}_{1,t}$ we take equation (2.16) and plug it back into the first block of (2.5) to give the expression,

$$\tilde{X}_{1,t} = S_{11}^{-1}T_{11}\tilde{X}_{1,t-1} + S_{11}^{-1}\left(\tilde{\Psi}_1 - \tilde{\Pi}_1\tilde{\Pi}_2^{-1}\tilde{\Psi}_2\right) z_t. \tag{2.17}$$

Even if there is more than one solution to (2.14) it is possible that they all lead to the same solution for $\tilde{X}_{1,t}$. Sims provides a second use of the singular value decomposition to check that the solution is unique. Equations (2.13) and (2.17) determine the evolution of $\left\{\tilde{X}_t\right\}$ as functions of the fundamental shocks $\{z_t\}$ and, using the definition of $\left\{\tilde{X}_t\right\}$ from (2.10), we can recover the original sequence $\{X_t\}$.

2.3 The Indeterminate Case

There are many examples of sensible economic models where the number of expectational variables is larger than the number of unstable roots of the system. In that case, Gensys will find a solution but flag the fact that there are many others. We propose to deal with that situation by providing a statistical model for one or more of the endogenous errors.

The rationale for our procedure is based on the notion that agents situated in an environment with multiple rational expectations equilibria must still choose to act. And to act rationally, they must form *some* forecast of the future and, therefore, we can model the process of expectations formation by specifying how the forecast errors covary with the other fundamentals.

If a model has n unstable generalized eigenvalues and p non-fundamental errors then, under some regularity assumptions, there will be $m = p - n$ degrees of indeterminacy. In that situation we propose to redefine m non-fundamental errors as new fundamental shocks. This transformation allows us to treat indeterminate models as determinate and to apply standard solution and estimation methods.

Consider model (2.1) and suppose that there are m degrees of indeterminacy. We propose to partition the η_t into two pieces, $\eta_{f,t}$ and $\eta_{n,t}$ and to partition Π conformably so that,

$$\underset{k\times k}{\Gamma_0}\underset{k\times 1}{X_t} = \underset{k\times k}{\Gamma_1}\underset{k\times 1}{X_{t-1}} + \underset{k\times l}{\Psi}\underset{l\times 1}{z_t} + \begin{bmatrix} \underset{k\times m}{\Pi_f} & \underset{k\times n}{\Pi_n} \end{bmatrix} \begin{bmatrix} \underset{m\times 1}{\eta_{f,t}} \\ \underset{n\times 1}{\eta_{n,t}} \end{bmatrix}. \tag{2.18}$$

Here, $\eta_{f,t}$ is an $m \times 1$ vector that contains the newly defined fundamental errors and $\eta_{n,t}$ contains the remaining n non-fundamental errors.

Next, we re-write the system by moving $\eta_{f,t}$ from the vector of expectational shocks to the vector of fundamental shocks:

$$\underset{k\times k}{\Gamma_0}\underset{k\times 1}{X_t} = \underset{k\times k}{\Gamma_1}\underset{k\times 1}{X_{t-1}} + \begin{bmatrix} \underset{k\times l}{\Psi} & \underset{k\times m}{\Pi_f} \end{bmatrix} \underset{(l+m)\times 1}{\widetilde{z}_t} + \underset{k\times n}{\Pi_n}\underset{n\times 1}{\eta_{n,t}}, \tag{2.19}$$

where we treat

$$\underset{(l+m)\times 1}{\widetilde{z}_t} = \begin{bmatrix} \underset{l\times 1}{z_t} \\ \underset{m\times 1}{\eta_{f,t}} \end{bmatrix}, \tag{2.20}$$

as a new vector of fundamental shocks and $\eta_{n,t}$ as a new vector of non-fundamental shocks.

To complete this specification, we define $\tilde{\Omega}$

$$\underset{(l+m)\times(l+m)}{\tilde{\Omega}} = E_{t-1}\left(\begin{bmatrix} \underset{l\times 1}{z_t} \\ \underset{m\times 1}{\eta_{f,t}} \end{bmatrix}\begin{bmatrix} \underset{l\times 1}{z_t} \\ \underset{m\times 1}{\eta_{f,t}} \end{bmatrix}^T\right) \equiv \begin{pmatrix} \underset{l\times l}{\Omega_{zz}} & \underset{l\times m}{\Omega_{zf}} \\ \underset{m\times l}{\Omega_{fz}} & \underset{m\times m}{\Omega_{ff}} \end{pmatrix}, \tag{2.21}$$

to be the new covariance matrix of fundamental shocks. This definition requires us to specify $m(m+1+2l)/2$ new variance parameters, these are the $m(m+1)/2$ elements of Ω_{ff}, and $m \times l$ new covariance parameters, these are the elements of Ω_{zf}. By choosing these new parameters and applying Sims' solution algorithm, we select a unique bounded rational expectations equilibrium. The diagonal elements of $\tilde{\Omega}$ that correspond to η_f have the interpretation of a pure 'sunspot' component to the shock and the covariance of these terms with z_t represent the response of beliefs to the original set of fundamentals.

Our approach to indeterminacy is equivalent to defining a new model in which the indeterminacy is resolved by assuming that expectations are formed consistently using the same forecasting method in every period. For example, expectations may be determined by a learning mechanism as in Evans and Honkapohja (2001) or using a belief function as in Farmer (2002). For our approach to be valid, we require that the belief function is time invariant and that shocks to that function can be described by a stationary probability distribution. Our newly transformed model can be written in the form of Equation (2.1), but the fundamental shocks in the transformed model include the original fundamental shocks z_t, as well as the vector of new fundamental shocks, $\eta_{f,t}$.

3 Choice of Expectational Errors

Our approach raises the practical question of which non-fundamentals should we choose to redefine as fundamental. Here we show that, given a relatively mild regularity condition, there is an equivalence between all possible ways of redefining the model.

Definition 1. [Regularity] Let ε be an indeterminate equilibrium of model (2.1) and use the QZ decomposition to write the following equation connecting fundamental and non-fundamental errors.

$$\tilde{\Psi}_2 z_t + \tilde{\Pi}_2 \eta_t = 0. \tag{3.1}$$

Let n be the number of generalized eigenvalues that are greater than or equal to 1 and let $p > n$ be the number of non-fundamental errors. Partition η_t into two mutually exclusive subsets, $\eta_{f,t}$ and $\eta_{n,t}$ such that $\eta_{f,t} \cup \eta_{n,t} = \eta_t$ and partition $\tilde{\Pi}_2$ conformably so that

$$\underset{n\times p}{\tilde{\Pi}_2}\underset{p\times 1}{\eta_t} = \begin{bmatrix} \underset{n\times m}{\tilde{\Pi}_{2f}} & \underset{n\times n}{\tilde{\Pi}_{2n}} \end{bmatrix} \begin{bmatrix} \underset{m\times 1}{\eta_{f,t}} \\ \underset{n\times 1}{\eta_{n,t}} \end{bmatrix}. \tag{3.2}$$

The indeterminate equilibrium, ε, is regular if, for all possible mutually exclusive partitions of η_t, $\tilde{\Pi}_{2n}$ has full rank.

Regularity rules out situations where there is a linear dependence in the non-fundamental errors and all of the indeterminate LRE models that we are aware of, that have been studied in the literature, satisfy this condition.

Theorem 1. *Let ε be an indeterminate equilibrium of model (2.1) and let $\mathbf{P}$ be an exhaustive set of mutually exclusive partitions of η_t into two non-intersecting subsets, where $\left\{\mathbf{p} \in \mathbf{P} \mid \mathbf{p} = \left(\underset{m\times 1}{\eta_{f,t}}, \underset{n\times 1}{\eta_{n,t}}\right)^T\right\}$. Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be elements of $\mathbf{P}$ and let $\tilde{\Omega}_1$ be the covariance matrix of the new set of fundamentals, $[z_t, \eta_{f,t}]$ associated with partition $\mathbf{p}_1$. If ε is regular then there is a covariance matrix $\tilde{\Omega}_2$, associated with partition 2 such that the covariance matrix*

$$\Omega = E\left(\begin{bmatrix} z_t \\ \eta_{f,t} \\ \eta_{n,t} \end{bmatrix}\begin{bmatrix} z_t \\ \eta_{f,t} \\ \eta_{n,t} \end{bmatrix}^T\right), \tag{3.3}$$

is the same for both partitions. $\mathbf{p}_1$ and $\mathbf{p}_2$, parameterized by $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$, are said to be equivalent partitions.

Proof. See Appendix A. □

Corollary 1. *The joint probability distribution over sequences* $\{X_t\}$ *is the same for all equivalent partitions.*

Proof. The proof follows immediately from the fact that the joint probability of sequences $\{X_t\}$, is determined by the joint distribution of the shocks. □

The question of how to choose a partition $\mathbf{p}_i$ is irrelevant since all partitions have the same likelihood. However, the partition *will* matter, if the researcher imposes zero restrictions on the variance covariance matrix of fundamentals.

Why does this matter? Suppose that the researcher choose one of two possible partitions, call this $\mathbf{p}_1$, by specifying one of two expectational errors from the original model as a new fundamental. Under partition $\mathbf{p}_1$, the covariance parameters of the second expectational error with the fundamentals will be complicated functions of all of the parameters of the model.

Suppose instead, that the researcher chooses the second expectational error to be fundamental, call this partition $\mathbf{p}_2$. In this case, it is the covariance parameters of the first expectational error that will depend on model parameters. Because the researcher cannot know in advance, which of these specifications is the correct one, we recommend that in practice, the VCV matrix of the augmented shocks, $\tilde{z}$, should be left unrestricted.

Lubik and Schorfheide (2004) refer to 'belief shocks' which they think of as independent causal disturbances that influence all of the endogenous variables at each date. Their belief shocks are isomorphic to what Cass and Shell (1983) refer to as 'sunspots' and what Azariadis (1981) and Farmer and Woodford (1984, 1997) call 'self-fulfilling prophecies'.

In Section 4, we prove that Lubik and Schorfheide's representation of a belief shock can be represented as a probability distribution over the forecast error of a subset of the variables of the model. Farmer (2002) shows how a self-fulfilling belief of this kind can be enforced by a forecasting rule, augmented by a sunspot shock. If agents use this rule in every period, and if their current beliefs about future prices are functions of the current sunspot shock, those beliefs will be validated in a rational expectations equilibrium.

4 Lubik-Schorfheide and Farmer-Khramov-Nicolò Compared

The two papers by Lubik and Schorfheide, (Lubik and Schorfheide, 2003, 2004), are widely cited in the literature (Belaygorod and Dueker, 2009; Zheng and Guo, 2013; Lubik and Matthes, 2013) and their approach is the one most closely emulated by researchers who wish to estimate models that possess an indeterminate equilibrium. This section compares the Lubik-Schorfheide method to the Farmer-Khramov-Nicolò technique (which we denote by LS and FKN) and proves an equivalence result.

We show in Theorem 2 that every LS equilibrium can be implemented as a FKN equilibrium, and conversely, every FKN equilibrium can be characterized using the LS technique. Because our method can be implemented using standard algorithms, our method provides an easy way for applied researchers to simulate and estimate indeterminate models using widely available computer software. And Theorem 2 shows that the full set of indeterminate equilibria can be modeled using our approach.

4.1 The Singular Value Decomposition

Determinacy boils down to the following question: Does equation (2.14), which we repeat below as equation (4.1), have a unique solution for the $p \times 1$ vector of endogenous errors, η_t, as functions of the $\ell \times 1$ vector of fundamental errors, z_t?

$$\underset{n\times\ell}{\tilde{\Psi}_2}\underset{\ell\times 1}{z_t} + \underset{n\times p}{\tilde{\Pi}_2}\underset{p\times 1}{\eta_t} = 0. \tag{4.1}$$

To answer this question, LS apply the singular value decomposition to the matrix $\tilde{\Pi}_2$. The interesting case is when $p > n$, for which $\tilde{\Pi}_2$ has n singular values, equal to the positive square roots of the eigenvalues of $\tilde{\Pi}_2 \tilde{\Pi}_2^T$. The singular values are collected into a diagonal matrix D_{11}. The matrices U_1 and V in the decomposition are orthonormal and $m = p - n$ is the degree of indeterminacy.

$$\underset{n\times p}{\tilde{\Pi}_2} \equiv \underset{n\times n}{U_1} \left[\begin{array}{cc} \underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}} \end{array} \right] \underset{p\times p}{V^T}. \tag{4.2}$$

Replacing $\tilde{\Pi}_2$ in (4.1) with this expression and premultiplying by U_1^T leads to the equation

$$\underset{n\times n}{U_1^T} \underset{n\times \ell}{\tilde{\Psi}_2} \underset{\ell\times 1}{z_t} + \left[\begin{array}{cc} \underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}} \end{array} \right] \underset{p\times p}{V^T} \underset{p\times 1}{\eta_t} = 0. \tag{4.3}$$

Now partition V

$$V = \left[\begin{array}{cc} \underset{p\times n}{V_1} & \underset{p\times m}{V_2} \end{array} \right],$$

and premultiply (4.3) by D_{11}^{-1},

$$\underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1^T} \underset{n\times \ell}{\tilde{\Psi}_2} \underset{\ell\times 1}{z_t} + \underset{n\times p}{V_1^T} \underset{p\times 1}{\eta_t} = 0. \tag{4.4}$$

Because $p > n$ this system has fewer equations than unknowns. LS suggest that we supplement it with the following new $m = p - n$ equations,

$$\underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{m\times m}{M_\zeta} \underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2^T} \underset{p\times 1}{\eta_t}. \tag{4.5}$$

The $m \times 1$ vector ζ_t is a set of sunspot shocks that is assumed to have mean zero and covariance matrix $\Omega_{\zeta\zeta}$ and to be uncorrelated with the fundamentals, z_t.

$$E[\zeta_t] = 0, \quad E\left[\zeta_t z_t^T\right] = 0, \quad E\left[\zeta_t \zeta_t^T\right] = \Omega_{\zeta\zeta}. \tag{4.6}$$

Correlation of the forecast errors, η_t, with fundamentals, z_t, is captured by the matrix M_z. Because the parameters of $\Omega_{\zeta\zeta}$ cannot separately be identified from the parameters of M_ζ, LS choose the normalization

$$M_\zeta = I_m. \tag{4.7}$$

Appending equation (4.5) as additional rows to equation (4.4), premultiplying by V and rearranging terms leads to the following representation of the expectational errors as functions of the fundamentals, z_t and the sunspot shocks, ζ_t,

$$\underset{p\times 1}{\eta_t} = \left(-\underset{p\times n}{V_1} \underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1^T} \underset{n\times \ell}{\tilde{\Psi}_2} + \underset{p\times m}{V_2} \underset{m\times \ell}{M_z} \right) \underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}. \tag{4.8}$$

This is equation (25) in Lubik and Schorfheide (2003) using our notation for dimensions and where our M_z is what LS call $\tilde{M}$. More compactly

$$\underset{p\times 1}{\eta_t} = \underset{p\times n}{V_1} \underset{n\times \ell}{N} \underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2} \underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}, \tag{4.9}$$

where

$$\underset{n\times \ell}{N} \equiv -\underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1^T} \underset{n\times \ell}{\tilde{\Psi}_2}.$$

is a function of the parameters of the model.

4.2 Equivalent characterizations of indeterminate equilibria

To define a unique sunspot equilibrium when the model is indeterminate, our method partitions η_t into two subsets; $\eta = \{\eta_f, \eta_n\}$. We refer to η_f as new fundamentals. A FKN equilibrium is characterized by a parameter vector $\theta \in \Theta_{FKN}$ which has two parts. $\theta_1 \in \Theta_1$

$$\theta_1 \equiv vec\left(\Gamma_0, \Gamma_1, \Psi, \Omega_z\right)^T,$$

is a vector of parameters of the structural equations, including the variance covariance matrix of the original fundamentals. And $\theta_2 \in \Theta_2$

$$\theta_2 \equiv vec\left(\Omega_{zf}, \Omega_{ff}\right)^T,$$

is a vector of parameters that contains the variance covariance matrix of the new fundamentals and the covariances of these new fundamentals, η_f, with the original fundamentals, z.

A FKN representation of equilibrium is a vector $\theta_{FKN} \in \Theta_{FKN}$ where Θ_{FKN} is defined as,

$$\Theta_{FKN} \equiv \{\Theta_1, \Theta_2\}.$$

Theorem 1 establishes that there is an equivalence class of models, all with the same likelihood function, in which the $m \times 1$ vector η_f is selected as a new set of fundamentals and the VCV matrices Ω_{ff} and Ω_{zf} are additional parameters. To complete the model in this way we must add $m\left(m+1\right)/2$ new parameters to define the symmetric matrix Ω_{ff} and $m \times \ell$ new parameters to define the elements of Ω_{zf}.

In contrast a LS equilibrium is characterized by a parameter vector

$$\Theta_{LS} \equiv \{\Theta_1, \Theta_3\},$$

where $\theta_3 \in \Theta_3$ is defined as

$$\theta_3 \equiv vec\left(\Omega_{\zeta\zeta}, M_z\right)^T. \tag{4.10}$$

These parameters characterize the additional equation,

$$\underset{m\times\ell}{M_z}\,\underset{\ell\times 1}{z_t} + \underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2^T}\,\underset{p\times 1}{\eta_t}, \tag{4.11}$$

where equation (4.11) adds the normalization (4.7) to equation (4.5).

The matrix $\Omega_{\zeta\zeta}$ has $m \times (m+1)/2$ new parameters; these are the variance covariances of the sunspot shocks and the matrix M_z has $m \times \ell$ new parameters, these capture the covariances of η with z. To establish the connection between the two characterizations of equilibrium, we establish the following two lemmas.

Proposition 1. *Let ε be a regular indeterminate equilibrium, characterized by $\theta_{FKN} = \{\theta_1, \theta_2\}$ and let $\mathbf{p}_i = \left\{\eta^i_{f,t}, \eta^i_{n,t}\right\}$ be an element of the set of partitions, $\mathbf{P}$. Let $\theta_{LS} = \{\theta_1, \theta_3\}$ be the parameters of a Lubik-Schorfheide representation of equilibrium. There is an $m \times m$ matrix G^i, and an $m \times \ell$ matrix H^i, where the elements of G^i and H^i, are functions of θ_1 and an $m \times \ell$ matrix S^i*

$$\underset{m\times\ell}{S^i} = \left(\underset{m\times\ell}{H^i} + \underset{m\times\ell}{M_z}\right), \tag{4.12}$$

such that the sunspots shocks in the LS representation of equilibrium are related to the fundamentals z_t and the newly defined FKN fundamentals, $\eta^i_{f,t}$ by the equation,

$$\underset{m\times 1}{\zeta_t} = \underset{m\times m}{G^i}\,\underset{m\times 1}{\eta^i_{f,t}} - \underset{m\times\ell}{S^i}\,\underset{\ell\times 1}{z_t}. \tag{4.13}$$

Proof. See Appendix B. □

Lemma 1 connects the LS sunspots to the FKN definition of fundamentals. Lemma 1, described below, provides a way of mapping between the original fundamental shocks and the newly defined fundamentals under two alternative partitions $\mathbf{p}_i$ and $\mathbf{p}_j$.

Lemma 1. *Let ε be a regular indeterminate equilibrium, characterized by $\theta_{FKN} = \{\theta_1, \theta_2\}$ and let $\mathbf{p}_i = \left\{\eta^i_{f,t}, \eta^i_{n,t}\right\}$ and $\mathbf{p}_j = \left\{\eta^j_{f,t}, \eta^j_{n,t}\right\}$ be two elements of the set of partitions, $\mathbf{P}$. There exists an $m \times m$ matrix G^i, an $m \times \ell$ matrix H^i, an $m \times m$ matrix G^j, and an $m \times \ell$ matrix H^j, where the elements of G^i, H^i, G^j and H^j are functions of θ_1. The new FKN fundamentals under partition $\mathbf{p}_i$, $\eta^i_{f,t}$, are related to the fundamentals z_t and the new FKN* *fundamentals under partition* $\mathbf{p}_j$, $\eta^j_{f,t}$ *by the equation,*

$$\underset{m\times 1}{\eta^i_{f,t}} = \underset{m\times m}{\left(G^i\right)^{-1}} \left[\underset{m\times m}{G^j} \underset{m\times 1}{\eta^j_{f,t}} - \left(\underset{m\times \ell}{H^j} - \underset{m\times \ell}{H^i} \right) \underset{\ell\times 1}{z_t} \right]. \tag{4.14}$$

Proof. Follows immediately from Equations (4.12) and (4.13) and the fact that G^i is non-singular for all i. □

Equation (4.14) defines the equivalence between alternative FKN definitions of the fundamental shocks, without reference to the LS definition. The following theorem, proved in Appendix C, uses Lemma 1 to establish an equivalence between the LS and FKN definitions.

Theorem 2. *Let* θ_{LS} *and* θ_{FKN} *be two alternative parameterizations of an indeterminate equilibrium in model (2.1). For every FKN equilibrium, parameterized by* θ_{FKN}*, there is a unique matrix* M_z *and a unique VCV matrix* $\Omega_{\zeta\zeta}$ *such that* $\theta_3 = vec\left(\Omega_{\zeta\zeta}, M_z\right)^T$ *and* $\{\theta_1, \theta_3\} \in \Theta_{LS}$ *defines an equivalent LS equilibrium. Conversely, for every LS equilibrium, parameterized by* θ_{LS}*, and every partition* $\mathbf{p}_i \in \mathbf{P}$*, there is a unique VCV matrix* Ω_{ff} *and a unique covariance matrix* Ω_{zf} *such that* $\theta_2 = vec(\Omega_{ff}, \Omega_{zf})^T$ *and* $\{\theta_1, \theta_2\} \in \Theta_{FKN}$ *defines an equivalent FKN equilibrium.*

Proof. See Appendix C. □

Next, we turn to an example that shows how to use our results in practice.

5 Applying Our Method in Practice: The Lubik-Schorfheide Example

In this Section we generate data from the model described in Lubik and Schorfheide (2004) and we use our method to recover parameter estimates from the simulated data. By using simulated data, rather than actual data, we avoid possible complications that might arise from mis-specification. For the simulated data, we know the true data generation process.

Section 5.1 explains how to implement our method for the case of the New-Keynesian model and in Section 5.2 we establish two results. First, we take Lubik and Schorfheide's (2004) parameter estimates for the pre-Volcker period, and we treat these parameter estimates as truth. Using the LS parameters, we simulate data under two alternative partitions of our model, and we verify that, using the same random seed, the simulated data are identical for both partitions. Second, we estimate the parameters of the model in Dynare, for the two alternative specifications, and we verify that the parameter estimates from two different partitions are the same.

5.1 The LS Model with the FKN Approach

The model of Lubik and Schorfheide (2004) consists of a dynamic IS curve

$$x_t = E_t\left(x_{t+1}\right) - \tau\left(R_t - E_t\left(\pi_{t+1}\right)\right) + g_t, \tag{5.1}$$

a New Keynesian Phillips curve

$$\pi_t = \beta E_t\left(\pi_{t+1}\right) + \kappa\left(x_t - z_t\right), \tag{5.2}$$

and a Taylor rule,

$$R_t = \rho_R R_{t-1} + \left(1-\rho_R\right)\left[\psi_1 \pi_t + \psi_2\left(x_t - z_t\right)\right] + \varepsilon_{R,t}. \tag{5.3}$$

The variable x_t represents log deviations of GDP from a trend path and π_t and R_t are log deviations from the steady state level of inflation and the nominal interest rate.

The shocks g_t and z_t follow univariate AR(1) processes

$$g_t = \rho_g g_{t-1} + \varepsilon_{g,t}, \tag{5.4}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t}, \tag{5.5}$$

where the standard deviations of the fundamental shocks $\varepsilon_{g,t}$, $\varepsilon_{z,t}$ and $\varepsilon_{R,t}$ are defined as σ_g, σ_z and σ_R, respectively. We allow the correlation between shocks, ρ_{gz}, ρ_{gR} and ρ_{zR}, to be nonzero. The rational expectation forecast errors are defined as

$$\eta_{1,t} = x_t - E_{t-1}\left[x_t\right], \quad \eta_{2,t} = \pi_t - E_{t-1}\left[\pi_t\right]. \tag{5.6}$$

We define the vector of endogenous variables,

$$X_t = \left[x_t, \pi_t, R_t, E_t\left(x_{t+1}\right), E_t\left(\pi_{t+1}\right), g_t, z_t\right]^T$$

the vectors of fundamental shocks and non-fundamental errors,

$$\mathbf{z}_t = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}\right]^T, \quad \eta_t = \left[\eta_{1,t}, \eta_{2,t}\right]^T$$

and the vector of parameters

$$\theta = \left[\psi_1, \psi_2, \rho_R, \beta, \kappa, \tau, \rho_g, \rho_z, \sigma_g, \sigma_z, \sigma_R, \rho_{gz}, \rho_{gR}, \rho_{zR}\right]^T.$$

This leads to the following representation of the model,

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\mathbf{z}_t + \Pi(\theta)\eta_t, \tag{5.7}$$

where Γ_0 and Γ_1 are represented by

$$\Gamma_0(\theta) = \begin{bmatrix} 1 & 0 & \tau & -1 & -\tau & -1 & 0 \\ \kappa & -1 & 0 & 0 & \beta & 0 & -\kappa \\ (1-\rho_R)\psi_2 & (1-\rho_R)\psi_1 & -1 & 0 & 0 & 0 & -(1-\rho_R)\psi_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and,

$$\Gamma_1(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho_R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_g & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_z \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

and the coefficients of the shock matrices Ψ and Π are given by,

$$\Psi(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Pi(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The last two rows of this system define the non-fundamental shocks and it is these rows that we modify when estimating the model with the FKN approach.

5.1.1 The Determinate Case

When the monetary policy is active, $|\psi_1| > 1$, the number of expectational variables, $\{E_t(x_{t+1}), E_t(\pi_{t+1})\}$, equals the number of unstable roots. The Blanchard-Kahn condition is satisfied and there is a unique sequence of non-fundamental shocks such that the state variables are bounded. In this case the model can be solved using Gensys which delivers the following system of equations

$$X_t = G_1(\theta)X_{t-1} + G_2(\theta)\mathbf{z}_t \tag{5.8}$$

where $G_1(\theta)$ represents the coefficients of the policy functions and $G_2(\theta)$ is the matrix which expresses the impact of fundamental errors on the variables of interest, X_t.

5.1.2 Indeterminate Models

A necessary condition for indeterminacy is that the monetary policy is passive, which occurs when

$$0 < |\psi_1| < 1. \tag{5.9}$$

A sufficient condition is that

$$0 < \psi_1 + \frac{(1-\beta)}{\kappa}\psi_2 < 1. \tag{5.10}$$

This condition is stronger than (5.9) but the two conditions are close, given our prior, which sets[3]

$$\frac{(1-\beta)}{\kappa}\psi_2 = 0.056.$$

[3] We thank one of the referees for pointing that the Taylor principle must be modified, when the central bank responds to the output gap as well as to inflation.

When (5.10) holds, the number of expectational variables, $\{E_t(x_{t+1}), E_t(\pi_{t+1})\}$, exceeds the number of unstable roots and there is 1 degree of indeterminacy. Using our approach, one can specify two equivalent alternative models depending on choice of the partition $\mathbf{p}_i$, for $i = 1, 2$.

Fundamental Output Expectations: Model 1 In our first specification, we choose $\eta_{1,t}$, the forecast error of output, as a new fundamental. We call this partition $\mathbf{p}_1$ and we write the new vector of fundamental shocks

$$\tilde{\mathbf{z}}_{1,t} = [\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \eta_{1,t}]^T .$$

The model is defined as

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi_x(\theta)\tilde{\mathbf{z}}_{1,t} + \Pi_x(\theta)\eta_{2,t}, \tag{5.11}$$

where

$$\Psi_x(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \Pi_x(\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Notice that the matrices Γ_0 and Γ_1 are unchanged. We have simply redefined $\eta_{1,t}$ as a fundamental shock by moving one of the columns of Π to Ψ. Because the Blanchard-Kahn condition is satisfied under this redefinition, the model can be solved using Gensys to generate policy functions as well as the matrix which describes the impact of the re-defined vector of fundamental shocks on X_t.

Fundamental Inflation Expectations: Model 2 Following the same logic there is an alternative partition $\mathbf{p}_2$ where the new vector of fundamentals is defined as

$$\tilde{\mathbf{z}}_{2,t} = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \eta_{2,t}\right]^T .$$

Here, the state equation is described by

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi_\pi(\theta)\tilde{\mathbf{z}}_{2,t} + \Pi_\pi(\theta)\eta_{1,t}, \tag{5.12}$$

where now

$$\Psi_\pi(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \Pi_\pi(\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Using Gensys, we can find a unique series of non-fundamental shocks $\eta_{1,t}$ such that the state variables are bounded and the state variables X_t are then a function of X_{t-1} and the new vector of fundamental errors $\tilde{\mathbf{z}}_{2,t}$.

5.2 Simulation and Estimation using the FKN approach

In this Section, we simulate data from the New-Keynesian model using the parameter estimates of Lubik and Schorfheide (2004) for the case when the model is indeterminate. In light of Theorem 2 and Lemma 1, data generated from the two partitions is identical, a result that we verify computationally. In Section 5.2.2, we use our simulated data to estimate the model parameters under the two representations and we confirm that the posterior modes from each representation are, in most cases, equal to two decimal places and that all of the estimates lie well within the 90% probability bounds of the alternative specification.[4] These results demonstrate how to apply our theoretical results from sections 3 and 4 in practice.

5.2.1 Simulation

In this section, we generate data for the observables, $\mathbf{y}_t = \{x_{obs,t}, \pi_{obs,t}, R_{obs,t}\}$, in two different ways. These variables are defined as,

1. $x_{obs,t}$ the percentage deviations of (log) real GDP per capita from an HP-trend;

2. $\pi_{obs,t}$ the annualized percentage change in the Consumer Price Index for all Urban Consumers;

3. $R_{obs,t}$ the annualized percentage average Federal Funds Rate.

As described in Lubik and Schorfheide (2004), the measurement equation is given by,

$$\mathbf{y}_t = \begin{bmatrix} 0 \\ \pi^* \\ \pi^* + r^* \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \end{bmatrix} X_t. \tag{5.13}$$

where π^* and r^* are annualized steady-state inflation and real interest rates expressed in percentages. The parameter values that we use to run the simulation of the New-Keynesian model in Lubik and Schorfheide (2004) are the posterior estimates that the authors report for the pre-Volcker period and that we reproduce in Table 2. We feed the model with shocks using the FKN method for two alternative partitions.

We take the LS estimates of the standard deviation of the sunspots shock, σ_ζ, and the $m \times \ell$ matrix M_z and we treat these estimates as the truth. By applying Lemma 1 to the LS

[4]The estimates are not identical because of sampling error that arises from the use of a finite number of draws when we approximate posterior distributions with the Metropolis-Hastings algorithm. We did not see an obvious way of setting the same random seed within Dynare and hence we used different draws for each specification.

parameters, we obtain corresponding values[5] for the standard deviation of the newly defined fundamental, $\eta_{f,t}^{i}$, under the two partitions, $\mathbf{p}_i$, $i \in \{1, 2\}$,

$$\underset{m\times m}{\Omega_{ff}^{i}} = \left(\underset{m\times m}{G^{i}}\right)^{-1}\left[\underset{m\times m}{\sigma_{\zeta}^{2}} + \underset{m\times \ell}{S^{i}}\underset{\ell\times\ell}{\Omega_{zz}}\underset{\ell\times m}{\left(S^{i}\right)^{T}}\right]\left(\underset{m\times m}{\left(G^{i}\right)^{T}}\right)^{-1}, \tag{5.14}$$

and for the covariance of the fundamentals $\mathbf{z}_t$ with the newly defined fundamental $\eta_{f,t}^{i}$,

$$\underset{m\times \ell}{\Omega_{fz}^{i}} = \underset{m\times m}{\left(G^{i}\right)^{-1}}\underset{m\times \ell}{S^{i}}\underset{\ell\times\ell}{\Omega_{zz}}. \tag{5.15}$$

The details on the construction of the matrices G^i, H^i and S^i are described in Appendix D.

Having defined the new vector of fundamentals $\tilde{\mathbf{z}}_{i,t} = \left[\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \eta_{i,t}\right]^T$ we construct the following variance-covariance matrix

$$\underset{(\ell+m)\times(\ell+m)}{\Omega^{i}} \equiv E\left(\tilde{\mathbf{z}}_{i,t}\ \tilde{\mathbf{z}}_{i,t}^{T}\right). \tag{5.16}$$

Next, we perform the Cholesky decomposition of the matrix $\Omega^i = L^i\left(L^i\right)^T$, where L^i is a lower triangular $(\ell+m)\times(\ell+m)$ matrix. After defining a $(\ell+m)\times 1$ vector of shocks u_t such that $E(u_t) = \mathbf{0}_{(\ell+m)\times 1}$ and $E(u_t u_t^T) = I_{(\ell+m)}$, we rewrite $\tilde{\mathbf{z}}_{i,t}$ as $\tilde{\mathbf{z}}_{i,t} = L^i u_t$.

The purpose of the Cholesky decomposition is to simplify the estimation procedure in Dynare[6] which we use to estimate the $(\ell+m)\times[(\ell+m)-1]$ parameters of the matrix L^i rather than the variance-covariance terms of the matrix Ω^i. Equation (5.17) reports the

[5] We derive both equation (5.14) and (5.15) from the result in Lemma 1 and by recalling that the vector of sunspot shocks ζ_t is now a scalar which, as described in Section 4.1, has the following properties, $E[\zeta_t] = 0$, $E\left[\zeta_t z_t^T\right] = 0$ and $E\left[\zeta_t \zeta_t^T\right] = \sigma_\zeta^2$.

[6] In particular, the estimation of the $(\ell+m)\times[(\ell+m)-1]$ elements of the lower triangular matrix L^i substantially reduces issues related to the convergence of the posterior estimates relative to the case of performing the estimation exercise by estimating the elements of the variance-covariance matrix Ω^i.

matrix Ω^i for $i = 1, 2$,

$$\Omega^1 = \begin{bmatrix} 0.05 & - & - & - \\ 0 & 0.07 & - & - \\ 0 & 0.04 & 1.27 & - \\ -0.03 & 0.10 & 0.11 & 0.17 \end{bmatrix}, \ \Omega^2 = \begin{bmatrix} 0.05 & - & - & - \\ 0 & 0.07 & - & - \\ 0 & 0.04 & 1.27 & - \\ -0.01 & 0.13 & -2.37 & 4.60 \end{bmatrix}, \tag{5.17}$$

and equation (5.18) is the corresponding Cholesky decomposition L^i for $i = 1, 2$,

$$L^1 = \begin{bmatrix} 0.23 & 0 & 0 & 0 \\ 0 & 0.27 & 0 & 0 \\ 0 & 0.15 & 1.11 & 0 \\ -0.14 & 0.37 & 0.04 & 0.10 \end{bmatrix}, \ L^2 = \begin{bmatrix} 0.23 & 0 & 0 & 0 \\ 0 & 0.27 & 0 & 0 \\ 0 & 0.15 & 1.11 & 0 \\ -0.05 & 0.04 & -2.12 & 0.26 \end{bmatrix}. \tag{5.18}$$

Given a draw of u_t, we obtain the new vector of fundamentals $\tilde{\mathbf{z}}_{i,t} = L^i u_t$ for partition $\mathbf{p}_i$ and we construct the corresponding draws of the vector $\tilde{\mathbf{z}}_{j,t} = [\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \eta_{j,t}]^T$. Using Lemma 1, Equation (4.14), which we reproduce below as equation (5.19), we derive the non-fundamental shock which is included as fundamental under partition $\mathbf{p}_j$ for $j \neq i$,

$$\underset{m\times 1}{\eta^j_{f,t}} = \underset{m\times m}{\left(G^j\right)^{-1}} \left[\underset{m\times m}{G^i} \underset{m\times 1}{\eta^i_{f,t}} - \left(\underset{m\times \ell}{H^i} - \underset{m\times \ell}{H^j} \right) \underset{\ell\times 1}{\mathbf{z}_t} \right]. \tag{5.19}$$

By feeding the two alternative models with the corresponding new vectors of fundamentals $\tilde{\mathbf{z}}_{1,t}$ and $\tilde{\mathbf{z}}_{2,t}$, using the same random seed, we obtain identical simulated data[7].

5.2.2 Estimation Results

Next, we estimate the parameters of the model on the simulated data and we demonstrate that the posterior estimates of the model parameters are equivalent under two alternative

[7]The code is available in the online Appendix and the results are obtained simulating the data by using both Gensys and Dynare.

model specifications. Table 1 reports the prior distributions of the parameters used in our estimation. With the exception of priors over the elements of L^i, the prior distributions for the other parameters are the same as in Lubik and Schorfheide (2004)[8].

Table 2 compares the posterior estimates of the model parameters. While the first column reports the parameter values used to simulate the data, columns two and three are the estimates for two alternative partitions $\mathbf{p}_1$ and $\mathbf{p}_2$. Partition $\mathbf{p}_1$ treats $\eta_{1,t}$ as fundamental and partition $\mathbf{p}_2$ treats $\eta_{2,t}$ as fundamental. We used a random walk Metropolis-Hastings algorithm to obtain 150,000 draws from the posterior mean and we report 90-percent probability intervals of the estimated parameters[9].

Compare the mean parameter estimates across the three columns. Fifteen of these parameters are common to all three specifications; these are the parameters $\psi_1, \psi_2, \rho_R, \pi^*, r^*, \kappa, \tau^{-1}, \rho_g, \rho_z, L_{11}, L_{22}, L_{33}, L_{21}, L_{31}$and L_{32}. The remaining four parameters reported in columns 2 and 3, $L^i_{41}, L^i_{42}, L^i_{43}$, and L^i_{44} represent the elements of the L^i matrix that are not comparable across specifications.

[8]The only difference with respect to Lubik and Schorfheide (2004) is that we use a flatter prior for the parameter κ. While the authors set a gamma distribution with mean 0.5 and standard deviation 0.2, our prior sets the standard deviation to 0.35, leaving the mean unchanged. Choosing a flatter prior avoids facing an issue in the convergence of the parameter which arises with a relatively tight prior as in Lubik and Schorfheide (2004).
Also, Table 1 reports the mean, the standard deviation and the 90-percent probability interval for each parameter. Note that we were unable to replicate the probability intervals in Lubik and Schorfheide (2004) and we report the 5-th and the 95-th percentiles of each distribution. However, the differences with Lubik and Schorfheide (2004) in the values for the probability intervals are small.

[9]To run the estimation exercise, we consider a sample of 1,000 observations from the simulated data, run 6 chains of 50,000 draws each and we finally discard half of the draws. The acceptance ratio for all the chains are between 25% and 33%.

Table 1: Prior Distribution for DSGE Model Parameters

Name	Range	Density	Mean	Std. Dev.	90% interval
ψ_1	$\mathbb{R}^+$	*Gamma*	1.1	0.50	[0.42,2.03]
ψ_2	$\mathbb{R}^+$	*Gamma*	0.25	0.15	[0.06,0.53]
ρ_R	$[0,1)$	*Beta*	0.50	0.20	[0.17,0.82]
π^*	$\mathbb{R}^+$	*Gamma*	4.00	2.00	[1.36,7.75]
r^*	$\mathbb{R}^+$	*Gamma*	2.00	1.00	[0.68,3.87]
κ	$\mathbb{R}^+$	*Gamma*	0.50	0.35	[0.09,1.17]
τ^{-1}	$\mathbb{R}^+$	*Gamma*	2.00	0.50	[1.25,2.88]
ρ_g	$[0,1)$	*Beta*	0.70	0.10	[0.54,0.85]
ρ_z	$[0,1)$	*Beta*	0.70	0.10	[0.54,0.85]
L_{11}	$\mathbb{R}^+$	*Inv.Gamma*	0.2	0.15	[0.07,0.44]
L_{22}	$\mathbb{R}^+$	*Inv.Gamma*	0.3	0.2	[0.12,0.64]
L_{33}	$\mathbb{R}^+$	*Inv.Gamma*	1	0.3	[0.61,1.55]
L_{21}		*Normal*	0	0.1	[-0.16,0.16]
L_{31}		*Normal*	0	0.1	[-0.16,0.16]
L_{32}		*Normal*	0.15	0.1	[-0.01,0.31]
L^1_{41}		*Normal*	0	0.2	[-0.32,0.32]
L^1_{42}		*Normal*	0.3	0.2	[-0.02,0.62]
L^1_{43}		*Normal*	0	0.2	[-0.32,0.32]
L^1_{44}		*Normal*	0.1	0.2	[-0.22,0.42]
L^2_{41}		*Normal*	0	0.2	[-0.32,0.32]
L^2_{42}		*Normal*	0	0.2	[-0.32,0.32]
L^2_{43}		*Normal*	-2	0.5	[-2.82,-1.18]
L^2_{44}		*Normal*	0.3	0.2	[-0.02,0.62]

Table 2: Posterior Means and Probability Intervals

L&S (prior 1)		FKN - Model 1		FKN - Model 2	
	Mean	Mean	90% interval	Mean	90% interval
ψ_1	0.77	0.77	[0.73,0.81]	0.77	[0.73,0.81]
ψ_2	0.17	0.21	[0.08,0.33]	0.22	[0.08,0.35]
ρ_R	0.60	0.61	[0.59,0.63]	0.61	[0.59,0.63]
π^*	4.28	4.44	[4.17,4.71]	4.43	[4.16,4.70]
r^*	1.13	1.18	[1.10,1.25]	1.17	[1.10,1.25]
κ	0.77	0.67	[0.47,0.89]	0.71	[0.51,0.91]
τ^{-1}	1.45	1.63	[1.41,1.85]	1.61	[1.39,1.82]
ρ_g	0.68	0.66	[0.62,0.70]	0.66	[0.62,0.70]
ρ_z	0.82	0.83	[0.81,0.84]	0.83	[0.81,0.85]
L_{11}	0.23	0.23	[0.22,0.24]	0.23	[0.22,0.24]
L_{22}	0.27	0.25	[0.21,0.29]	0.25	[0.21,0.29]
L_{33}	1.11	1.14	[0.90,1.37]	1.10	[0.87,1.30]
L_{21}	0	-0.01	[-0.03,0.009]	-0.01	[-0.03,0.009]
L_{31}	0	0.02	[-0.09,0.14]	0.003	[-0.09,0.09]
L_{32}	0.15	0.14	[0.01,0.27]	0.14	[0.04,0.25]
L^1_{41}	-0.14	-0.15	[-0.18,-0.13]	-	-
L^1_{42}	0.37	0.36	[0.34,0.37]	-	-
L^1_{43}	0.04	0.02	[-0.02,0.07]	-	-
L^1_{44}	0.10	0.10	[-0.20,0.42]	-	-
L^2_{41}	-0.05	-	-	-0.07	[-0.25,0.11]
L^2_{42}	0.04	-	-	0.03	[-0.17,0.22]
L^2_{43}	-2.12	-	-	-2.09	[-2.16,-2.01]
L^2_{44}	0.26	-	-	0.30	[-0.02,0.62]

Our results show not only that under both models the posterior point estimates are remarkably close to the parameter values which we use to simulate the data, but also that both the posterior point estimates and the probability intervals are statistically indistinguishable when comparing the two alternative models. This correspondence in parameter estimates across specifications is a consequence of Theorems 1 and 2 of our paper.

6 Implementing our Procedure in Dynare

This section provides a practical guide to the user who wishes to implement our method in Dynare. Consider the New-Keynesian model described in Section 5, which we repeat below for completeness,

$$x_t = E_t[x_{t+1}] - \tau(R_t - E_t[\pi_{t+1}]) + g_t, \tag{6.1}$$

$$\pi_t = \beta E_t[\pi_{t+1}] + \kappa x_t + z_t, \tag{6.2}$$

$$g_t = \rho_g g_{t-1} + \varepsilon_{g,t}, \tag{6.3}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t}, \tag{6.4}$$

The model is determinate when monetary policy is active, $|\psi_1| > 1$. In this case Dynare finds the unique series of non-fundamental errors that keeps the state variables bounded and Table 3 reports the code required to estimate the model in this case.

In the case of the indeterminate models described in Section 5.1.2, running Dynare with the code from Table 3 produces an error with a message "Blanchard-Kahn conditions are not satisfied: indeterminacy." For regions of the parameter space where the code produces that message, we provide two alternative versions of the model that redefine one of the non-fundamental shocks as new fundamental. Following the notation in Section 5.1.2, we refer to these cases as Model 1, where $\eta_{1,t} = x_t - E_{t-1}[x_t]$ is a fundamental shock, and Model 2, where it is $\eta_{2,t} = \pi_t - E_{t-1}[\pi_t]$ and we present the Dynare code to estimate the two indeterminate cases.

Table 3: **Determinate Model**

Variable Definitions	var x, R, pi, g, z; varexo e_R, e_g, e_z;
Parameter Definitions	parameters $tau, kappa, rho\_R, rho\_g, rho\_z, psi1$ $psi2$;

Model equations	model(linear);
	$x = x\,(+1) - tau * (R - pi\,(+1)) + g$;
	$pi = 0.97 * pi\,(+1) + kappa * (x - z)$;
	$R = rho\_R * R\,(-1) + (1 - rho\_R) *$
	$(psi1 * pi + psi2 * (x - z)) + e\_R$;
	$g = rho\_g * g(-1) + e\_g$;
	$z = rho\_z * z(-1) + e\_z$;
	end;

Tables 4 and 5 present the amended code for these cases. In Table 4, we show how to change the model by redefining $\eta_{1,t}$ as fundamental and Table 5 presents an equivalent change to Table 3 in which $\eta_{2,t}$ becomes the new fundamental. We have represented the new variables and new equations in that table using bold typeface.

The following steps explain the changes in more detail. First, we define a new variable, $\mathbf{xs} \equiv E_t [x_{t+1}]$ and include it as one of the endogenous variables in the model. This leads to the declaration:

$$var\ x,\ R,\ pi,\ \mathbf{xs}; \tag{6.5}$$

which appears in the first line of Table 4. Next, we add an expectational shock, which we call **sunspot**, to the set of fundamental shocks, e_R, e_g and e_z. This leads to the Dynare statement

$$varexo\ e\_R,\ e\_g,\ e\_z,\ \mathbf{sunspot}; \tag{6.6}$$

which appears in row 2. Then we replace $x\,(+1)$ by **xs** in the consumption-Euler equation, which becomes,

$$x = \mathbf{xs} - tau * (R - pi\,(+1)) + g; \tag{6.7}$$

and we add a new equation that defines the relationship between **xs**, x and the new fundamental error:

$$x - \mathbf{xs}\,(-1) = \mathbf{sunspot}; \tag{6.8}$$

Table 4: **Indeterminate Model 1:** $\eta_{1,t} = x_t - E_{t-1}[x_t]$ is new fundamental

Variable Definitions	var $x, R, pi, g, z,$ **xs**; varexo $e\_R, e\_g, e\_z,$ **sunspot**;
Parameter Definitions	parameters $tau, kappa, rho\_R, rho\_g, rho\_z, psi1$ $psi2, sigmag, sigmaz, sigmaR$;

Model equations	model(linear);
	$x = \mathbf{xs} - tau * (R - pi\,(+1)) + g;$
	$pi = 0.97 * pi\,(+1) + kappa * (x - z);$
	$R = rho\_R * R\,(-1) + (1 - rho\_R) *$
	$(psi1 * pi + psi2 * (x - z)) + e\_R;$
	$g = rho\_g * g(-1) + e\_g;$
	$z = rho\_z * z(-1) + e\_z;$
	$x - \mathbf{xs}\,(-1) = \mathbf{sunspot};$
	end;

Similar steps apply in the case of Model 2, but with $\eta_{2,t}$ taking the role of $\eta_{1,t}$. Note that, by substituting expectations of forward-looking variables $x(+1)$ in Model 1, and $pi(+1)$ in Model 2, with **xs** and **pis**, respectively, we decrease the number of forward-looking variables by one. Since these variables are no longer solved forwards, we must add an equation – this appears as Equation (6.8) – to describe the dynamics of the new fundamental shock.

Table 5: **Indeterminate Model 2:** $\eta_{2,t} = \pi_t - E_{t-1}[\pi_t]$ is new fundamental

Variable Definitions	var $x, R, pi, g, z,$ **pis**; varexo $e\_R, e\_g, e\_z,$ **sunspot**;
Parameter Definitions	parameters $tau, kappa, rho\_R, rho\_g, rho\_z, psi1$ $psi2, sigmag, sigmaz, sigmaR$;

Model equations	model(linear);
	$x = x(+1) - tau * (R - pi\,(+1)) + g$;
	$pi = 0.97 * \mathbf{pis} + kappa * (x - z)$;
	$R = rho\_R * R\,(-1) + (1 - rho\_R) *$
	$(psi1 * pi + psi2 * (x - z)) + e\_R$;
	$g = rho\_g * g(-1) + e\_g$;
	$z = rho\_z * z(-1) + e\_z$;
	$pi - \mathbf{pis}\,(-1) = \mathbf{sunspot}$;
	end;

How can a researcher know, in advance, if his model is determinate. The answer provided by Lubik and Schorfheide (2004), is that determinate and indeterminate models are alternative representations of data that can be compared either by Likelihood ratio tests or by Bayesian model comparison.

The Lubik-Schorfheide approach assumes that the researcher can identify, a priori, determinate and indeterminate regions of the parameter space. For models where that is difficult or impossible, Fanelli (2012) and Castelnuovo and Fanelli (2014) propose an alternative method that may be used to test the null hypothesis of determinacy.

7 Conclusion

Our paper provides a method to solve and estimate indeterminate linear rational expectations models using standard software packages. Our method transforms indeterminate models by redefining a subset of the non-fundamental shocks and classifying them as new fundamentals. Our approach to handling indeterminate equilibria is more easily implementable than that of Lubik and Schorfheide and, one might argue, is also more intuitive. We illustrated our approach using the familiar New-Keynesian monetary model and we showed that, when monetary policy is passive, the new-Keynesian model can be closed in one of two equivalent ways.

Our procedure raises the question of which non-fundamental shocks to reclassify as fundamental. Our theoretical results demonstrate that the choice of parameterization is irrelevant since all parameterizations have the same likelihood function. We demonstrated that result in practice by estimating a model due to Lubik and Schorfheide (2004) in two different ways and recovering parameter estimates that are statistically indistinguishable between the two. We caution that, in practice, it is important to leave the VCV matrix of errors unrestricted for our results to apply. Our work should be of interest to economists who are interested in estimating models that do not impose a determinacy prior.

Part II

A Generalized Approach to Indeterminacy in Linear Rational Expectations Models

Sunspot shocks and multiple equilibria have been at the center of economic thinking at least since the seminal work of Cass and Shell (1983), Farmer and Guo (1994) and Farmer and Guo (1995). Furthermore, in many of the Linear Rational Expectation (LRE) models used to study the properties of the macroeconomy the possibility of multiple equilibria arises for some parameter values, but not for others. This paper proposes a novel approach to solve LRE models that easily accommodates both the case of determinacy and indeterminacy. As a result, the proposed methodology can be used to easily estimate a LRE model that could potentially be characterized by multiplicity of equilibria. Our approach is implementable even when the analytic conditions for determinacy or the degrees of indeterminacy are unknown and can be implemented to study indeterminacy in standard software packages, such as Dynare and Sims' (2001) code Gensys.

To understand how our approach works, it is useful to recall the conditions for determinacy as stated by Blanchard and Kahn (1980a). Indeterminacy arises when the parameter values are such that the number of explosive roots is smaller than the number of non-predetermined variables. The key idea behind our methodology consists of augmenting the original model by appending additional autoregressive processes that can be used to provide the missing explosive roots. The innovations of these exogenous processes are assumed to be linear combinations of a subset of the forecast errors associated with the expectational variables of the model, and a newly defined vector of sunspot shocks. Whether the autoregressive processes are mean-reverting or explosive is central, and the intuition follows. When a model is determinate, the roots of the additional autoregressive process are within the unit circle (i.e., the Blanchard-Kahn condition is satisfied) and the auxiliary process is irrelevant for the dynamics of the model. The law of motion for the endogenous variables is in this case equivalent to the solution obtained using standard solution algorithms (King and Watson (1998), Klein (2000), Sims (2001b)). When the model is indeterminate, the appended autoregressive processes are explosive, and the solution we obtain for the endogenous variables is equivalent to the one obtained with the methodology of Lubik and Schorfheide (2003) or, equivalently, Farmer et al. (2015).

Our methodology simplifies the common approach used to deal with indeterminacy. First, the common procedure requires the researcher to solve the model differently depending on the area of the parameter space that is being studied. Second, the procedure requires to estimate the same model twice, first under determinacy, then under indeterminacy. This is the same procedure that would be followed if the researcher were comparing two *structurally different* models, while she is in fact estimating the *same structural* model in alternative regions of the parameter space. Finally, the estimation under indeterminacy is not generally implementable in standard estimation packages and requires a significant amount of coding work on the side of the researcher.

In this respect, our methodology provides three main advantages. First, it accommodates both the case of determinacy and indeterminacy while considering the same augmented system of equations. The model can therefore be solved by using standard solution algorithms. Instead, existing methods require to rewrite the model based on the existing degree of indeterminacy (Farmer et al. (2015)) or to construct the solution under indeterminacy ex-post following the seminal contribution of Lubik and Schorfheide (2003). Second, given that the method accommodates both the case of determinacy and indeterminacy, the researcher does not need to take a stance on which area of the parameter space she is interested in exploring. We show that our methodology ensures that standard estimation algorithms explore the entire parameter space, increasing the probability of finding a global maximum over the parameter space. This is particularly relevant when considering that the posterior mode is a crucial object used for Bayesian inference.[10] Finally, even when the region of determinacy is unknown, the methodology allows the researcher to estimate the model without imposing *a priori* assumptions about the uniqueness of the equilibrium, which can be equivalently thought of as restrictions on the parameter space over which inference is conducted. Hence, information contained in the data indicates whether an estimated model is characterized by a unique solution or by multiplicity of equilibria.

Our work is related to the vast literature that studies the role of indeterminacy in explaining the evolution of the macroeconomy. Prominent examples in the monetary policy literature include the work of Clarida et al. (2000b) and Kerr and King (1996), that study the possibility of multiple equilibria as a result of violations of the Taylor principle in New-Keynesian (NK) models. Applying the methods developed in Lubik and Schorfheide (2003) to the canonical NK model, Lubik and Schorfheide (2004) test for indeterminacy in U.S. monetary policy. Using a calibrated small-scale model, Coibon and Gorodnichenko (2011) find that the reduction of the target inflation rate in the U.S. also played a key role in explaining the Great Moderation, and Arias et al. (2017) support this finding in the context of a medium-scale model *à la* Christiano et al. (2005). In a similar spirit, Arias (2013) studies the dynamic properties of medium-sized NK models with trend inflation. More recently, Aruoba and Schorfheide (2015) study inflation dynamics at the Zero Lower Bound (ZLB) and during an exit from the ZLB.

The paper closest to our is Farmer et al. (2015). As explained above, the main difference between the two approaches is that our method accommodates both the case of determinacy and indeterminacy while considering the same augmented system of equations. Instead, the

[10]Specifically, using simulated data we show that our methodology leads the estimation algorithm to converge to the "right" area of the parameter space. Once the algorithm converges to such area, the probability of leaving it is very low, in line with the results of Lubik and Schorfheide (2004) that show that the likelihood presents potentially very large jumps/drops between the determinacy/indeterminacy regions.

method proposed by Farmer et al. (2015) require to rewrite the model based on the existing degree of indeterminacy. With respect to Lubik and Schorfheide (2003), the main novelty of our approach is to provide a unified approach to study determinacy and indeterminacy of different degrees. Ascari et al. (2016) allow for temporarily unstable paths, while we require all solutions to be stationary, in line with previous contributions in the literature. Finally, we deliberately use Dynare in all the examples presented in this paper to show that our method can be combined with standard packages. However, our solution method can be combined with more sophisticated estimation techniques such as the ones developed in Herbst and Schorfheide (2015).

The remainder of the paper is organized as follows. Section 1 builds the intuition by using a univariate example in the spirit of Lubik and Schorfheide (2004). Section 2 describes the methodology and shows that the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) and Farmer et al. (2015). Section 3 provides an analytic example of the theoretical result and in Section 4, we apply our theoretical results to the NK model of Lubik and Schorfheide (2004). In Section 4.1, we first generate series of simulated data for parameter values which satisfy the condition for determinacy and indeterminacy, respectively. We then estimate the model by using the proposed augmented representation for both cases. The model is estimated over the entire parameter space and the true parameter values are recovered, providing evidence in favor of determinacy or indeterminacy. Section 4.2 shows that this is true even when we assume that the researcher does not know the boundaries of the determinacy region. Hence, our methodology can be used to test for indeterminacy in a wide class of models, including medium- and large-scale models for which the region of determinacy cannot be derived analytically. We also repeat the exercise on actual data using the dataset from Lubik and Schorfheide (2004) in Section 4.3. Finally, we provide guidance on how to properly implement our methodology in Section

5 and our conclusions are presented in Section 6.

1 Building the intuition

Before presenting the theoretical results of the paper, this section builds the intuition behind our approach by considering a univariate example similar to the one proposed in Lubik and Schorfheide (2004). While Section 1.1 explains our approach from an analytical perspective, Section 1.2 addresses questions which could arise at the time of its practical implementation.

1.1 A useful example

Consider a classical monetary model characterized by the Fisher equation

$$i_t = E_t(\pi_{t+1}) + r_t, \tag{1.1}$$

and the simple Taylor rule

$$i_t = \phi_\pi \pi_t, \tag{1.2}$$

where i_t denotes the nominal interest rate. We assume that the real interest rate r_t is given and described by a mean-zero Gaussian i.i.d. shock.[11] To properly specify the model, we also define the one-step ahead forecast error associated with the expectational variable, π_t, as

$$\eta_t \equiv \pi_t - E_{t-1}(\pi_t). \tag{1.3}$$

[11] In the classical monetary model, the real interest rate results from the equilibrium in labor and goods market and it depends on the technology shocks. We are considering an exogenous process for the technology shocks and therefore we take the process for the real interest rate as given.

Combining (1.1) and (1.2), we obtain the univariate model

$$E_t(\pi_{t+1}) = \phi_\pi \pi_t - r_t. \tag{1.4}$$

First, we consider the case $\phi_\pi > 1$. Rewriting equation (1.4), it is clear that this case is associated with the determinate solution,

$$\begin{aligned} \pi_t &= \frac{1}{\phi_\pi} E_t(\pi_{t+1}) + \frac{1}{\phi_\pi} r_t \qquad (1.5) \\ &= \frac{1}{\phi_\pi} r_t. \qquad (1.6) \end{aligned}$$

where the last equality is obtained by solving equation (1.5) forward and recalling the assumptions on r_t. The strong response of the monetary authority to changes in inflation ($\phi_\pi > 1$) guarantees that inflation is pinned down as a function of the exogenous real interest r_t. From a technical perspective, the condition $\phi_\pi > 1$ is such that the Blanchard-Kahn condition is satisfied: the number of explosive roots matches the number of expectational variables, that in this univariate case is π_t.

The second case corresponds to $\phi_\pi \leq 1$. The solution to (1.4) is obtained by combining (1.4) with (1.3), and it corresponds to any process that takes the following form

$$\pi_t = \phi_\pi \pi_{t-1} - r_{t-1} + \eta_t. \tag{1.7}$$

When the monetary authority does not respond aggressively enough to changes in inflation ($\phi_\pi \leq 1$), there are multiple solutions for the inflation rate, π_t, each indexed by the expectations that the representative agent holds about future inflation, η_t. Equivalently, the solution to the univariate model is indeterminate: the Blanchard-Kahn solution is not satisfied since there is no explosive root to match the number of expectational variables, that is π_t.

From a methodological and computational perspective, the latter case constitutes a challenge. Standard software packages such as Dynare do not allow for indeterminacy. Of course, a researcher could in principle code an estimation algorithm herself, following the methods outlined in Lubik and Schorfheide (2004). However, this approach requires a substantial amount of time and technical skills. Hence, the result is that in practice most of the papers simply rule out the possibility of indeterminacy, even if the model at hand could in principle allow for such a feature.

From a purely technical point of view, the problem that a researcher faces when solving a LRE model such as the one presented in (1.4) using standard solution algorithms is the following. Under determinacy, the model already has a sufficient number of unstable roots to match the number of expectational variables. However, under indeterminacy, the model is missing one explosive root since it still has one expectational variable, but no unstable root. Therefore, our approach proposes to augment the original model by appending an independent process which could be either stable or unstable. The key insight consists of choosing this auxiliary process in a way to deliver the correct solution. When the original model is determinate, the auxiliary process must be stationary so that also the augmented representation satisfies the Blanchard-Kahn condition. In this case, the auxiliary process represents a separate block that does not affect the law of motion of the model variables. When the model is indeterminate, the additional process should however be explosive so that the Blanchard-Kahn condition is satisfied for the augmented system, even if not for the original model. In what follows, we apply this intuition to the example considered in this section and explain how to choose the auxiliary process.

Our methodology proposes to solve an augmented system of equations which can be dealt with by using standard solution algorithms such as Sims (2001b) under both determinacy

Blanchard-Kahn condition in the augmented representation			
	Unstable Roots	B-K condition in augmented model (1.8)	Solution
Determinacy $\phi_\pi > 1$ in original model (1.4)			
$\frac{1}{\alpha} < 1$	1	Satisfied	$\left\{\pi_t = \frac{1}{\phi_\pi} r_t,\ \omega_t = \alpha\omega_{t-1} - \nu_t + \varepsilon_t\right\}$
$\frac{1}{\alpha} > 1$	2	Not satisfied	-
Indeterminacy $\phi_\pi \leq 1$ in original model (1.4)			
$\frac{1}{\alpha} < 1$	0	Not satisfied	-
$\frac{1}{\alpha} > 1$	1	Satisfied	$\{\pi_t = \phi_\pi\pi_{t-1} - r_{t-1} + \eta_t, \omega_t = 0\}$

Table 1.1: Regions of the parameter space for which the Blanchard-Kahn condition in the augmented representation is satisfied, even when the original model is indeterminate.

and indeterminacy. Consider the following augmented system

$$\begin{cases} E_t(\pi_{t+1}) = \phi_\pi \pi_t - r_t, \\ \omega_t = \left(\frac{1}{\alpha}\right)\omega_{t-1} - \nu_t + \eta_t, \end{cases} \tag{1.8}$$

where ω_t is an independent autoregressive process, $\alpha \in [0, 2]$ and ν_t is a newly defined mean-zero sunspot shock with standard deviation σ_v. Table 1.1 summarizes the intuition behind our approach.

When the original LRE model in (1.4) is determinate, $\phi_\pi > 1$, the Blanchard-Kahn condition for the augmented representation in (1.8) is satisfied when $1/\alpha < 1$. Indeed, for $\phi_\pi > 1$ the original model has the same number of unstable roots as the number of expectational variables. Our methodology thus suggests to append a stable autoregressive process. When $\phi_\pi > 1$ and $1/\alpha < 1$, the solution method of Sims (2001b) delivers the same solution for the endogenous variable π_t as in equation (1.5). Since the coefficient $1/\alpha$ is smaller than 1, the solution also includes the autoregressive process ω_t. Importantly, its dynamics do not impact the endogenous variable y_t.

Considering the case of indeterminacy (i.e. $\phi_\pi \leq 1$), the original model has one expectational variable, but no unstable root, thus violating the Blanchard-Kahn condition. By appending an explosive autoregressive process, the augmented representation that we propose satisfies the Blanchard-Kahn condition and delivers the same solution as the one resulting from the methodology of Lubik and Schorfheide (2003) or Farmer et al. (2015) described by equation (1.7). Moreover, stability imposes conditions such that ω_t is always equal to zero, and the solution for the endogenous variable does not depend on the appended autoregressive process.

Summarizing, the choice of the coefficient $\frac{1}{\alpha}$ should be made as follows. For values of ϕ_π greater than 1, the Blanchard-Kahn condition for the augmented representation is satisfied for values of α greater than 1. Conversely, under indeterminacy (i.e. $\phi_\pi \leq 1$) the condition is satisfied when α is smaller than 1. The choice of parametrizing the auxiliary process with $1/\alpha$ instead of α induces a positive correlation between ϕ_π and α that facilitates the implementation of our method when estimating a model.

Finally, note that under both determinacy and indeterminacy, the exact value of $1/\alpha$ is irrelevant for the law of motion of π_t. Under determinacy, the auxiliary process ω_t is stationary, but its evolution does not affect the law of motion of the model variables. Under indeterminacy, ω_t is always equal to zero. This makes clear that introducing the auxiliary processes does not affect the properties of the solution in the two cases. These processes only serve the purpose of providing the necessary explosive roots under indeterminacy and of creating the mapping between the sunspot shocks and the expectation errors. As we will see in Section 2, this result can be generalized and applies to more complicated models with potentially multiple degrees of indeterminacy.

1.2 Choosing α

A natural question that arises with the approach we propose is how to choose α. We consider the following three cases: (1) The researcher knows the analytic condition defining the region of determinacy; (2) she only has an relatively good idea of the threshold of the determinacy region; (3) the region of determinacy is completely unknown to the researcher. We consider the three cases separately.

We first consider the case in which the researcher is able to analytically derive the condition which defines when the model is determinate or indeterminate. For the example considered in this section, this case corresponds to knowing that when $\phi_\pi \leq 1$ the model in (1.4) is indeterminate. We thus suggest to write the parameter α as a function of the parameter ϕ_π so that the augmented representation in (1.8) always satisfies the Blanchard-Kahn condition. This can be obtained by setting $\alpha \equiv \phi_\pi$. When the original model is determinate ($\phi_\pi > 1$), the appended autoregressive process is stationary since $1/\alpha < 1$. Under indeterminacy ($\phi_\pi \leq 1$) of the original model, the coefficient $1/\alpha$ is greater than 1 and the appended process is therefore explosive. Hence, when the region of determinacy is known, the researcher should carefully choose α such that the augmented representation always delivers a solution under both determinacy and indeterminacy. Using the NK model in Lubik and Schorfheide (2004), we implement this suggestion in Section 4.1 where we estimate the model assuming that the researcher knows the region of determinacy.

There are however instances in which the researcher does not know the exact properties of the determinacy region. Suppose that the researcher does not know the region, but for values of the parameter ϕ_π slightly above 1 she can solve the original model under determinacy, while for values just below 1 the model is indeterminate. She thus has a relatively good idea that the threshold for determinacy is somewhere around 1, even if she is not able to derive the analytical condition. In this case, she could set a prior distribution for the parameter α such that there is a higher probability of drawing values above 1 when the parameter ϕ_π is greater than 1 and vice versa. Similarly, the variance-covariance matrix used by the Metropolis-Hastings algorithm to propose new draws should be chosen to display a positive correlation between the values of ϕ_π and α. This practice would increase the likelihood of obtaining a solution in the augmented representation and therefore the efficiency of the algorithm.

Finally, it could be the case that the region of determinacy is completely unknown to the researcher. For a given draw of the parameter ϕ_π, the researcher would like to make draws of α smaller or greater than 1 with equal probabilities. In this case, the researcher could use a uniform distribution over the interval $[0, 2]$ or any symmetric interval around 1 as a prior distribution. Also, note that the prior distribution does not necessarily have to be continuous. A discrete probability distribution that allows to make draws of α to be either equal 0.5 or 1.5 could also be specified as a prior. In this context, the efficiency of the algorithm would also be improved if it were to be designed as follows. If for a given draw of ϕ_π and α the augmented representation in (1.8) does not have a solution, the algorithm should be coded as to make a new draw of α' equal to the inverse of the earlier draw α.

2 Methodology

Given the general class of LRE models described in Sims (2001b), this paper proposes an augmented representation which embeds the solution for the model under both determinacy and indeterminacy.[12] In particular, the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) or equivalently Farmer et al. (2015). In the following, we generalize the intuition built in the previous section. Consider the following LRE model

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t, \tag{2.1}$$

where $X_t \in R^k$ is a vector of endogenous variables, $\varepsilon_t \in R^\ell$ is a vector of exogenous shocks, $\eta_t \in R^p$ collects the one-step ahead forecast errors for the expectational variables of the system and $\theta \in \Theta$ is a vector of parameters. The matrices Γ_0 and Γ_1 are of dimension $k \times k$,

[12] In this paper we focus on linear rational expectation models. Schmitt-Grohé and Uribe (2004) and Judd (1998) use perturbation methods to solve DSGE models using higher order approximations.

possibly singular, and the matrices Ψ and Π are of dimension $k \times \ell$ and $k \times p$, respectively. Also, we assume

$$E_{t-1}(\varepsilon_t) = 0, \quad \text{and} \quad E_{t-1}(\eta_t) = 0. \tag{2.2}$$

We also define the $\ell \times \ell$ matrix $\Omega_{\varepsilon\varepsilon}$,

$$\Omega_{\varepsilon\varepsilon} \equiv E_{t-1}(\varepsilon_t \varepsilon_t^T), \tag{2.3}$$

which represents the covariance matrix of the exogenous shocks.

Consider a model whose maximum degree of indeterminacy is denoted by m.[13] The proposed methodology appends to the original LRE model in (2.1) the following system of m equations

$$\omega_t = \Phi \omega_{t-1} + \nu_t - \eta_{f,t}, \qquad \Phi \equiv \begin{bmatrix} \frac{1}{\alpha_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{1}{\alpha_m} \end{bmatrix} \tag{2.4}$$

where the vector $\eta_{f,t}$ is a subset of the endogenous shocks and the vectors $\{\omega_t, \nu_t, \eta_{f,t}\}$ are of dimension $m \times 1$. The equations in (2.4) are autoregressive processes whose innovations are linear combinations of a vector of sunspot shocks, ν_t, and a subset of forecast errors, $\eta_{f,t}$, where $E_{t-1}(\nu_t) = E_{t-1}(\eta_{f,t}) = 0$. As we will show below, the choice of which expectational errors to include in (2.4) does not affect the solution.

The intuition behind the proposed methodology works as in the example considered in the previous section. Let $m^*(\theta)$ denote the actual degree of indeterminacy associated with the parameter vector θ. Under indeterminacy the Blanchard-Kahn condition for the original LRE model in (2.1) is not satisfied. Suppose that the system is characterized by $m^*(\theta)$

[13]Denoting by n the minimum number of unstable roots of a LRE model, the maximum degrees of indeterminacy are defined as $m \equiv p - n$. When the minimum number of unstable roots of a model is unknown, then m coincides with number of expectational variables p. This represents the maximum degree of indeterminacy in any model with p expectational variables.

degrees of indeterminacy, then it is necessary to introduce $m^*(\theta)$ explosive roots to solve the model using standard solution algorithms. In this case, the (absolute value of the) $m^*(\theta)$ of the diagonal elements of the matrix Φ are assumed to be outside the unit circle, and the augmented representation is therefore determinate since the Blanchard-Kahn condition is now satisfied. On the other hand, under determinacy the (absolute value of the) diagonal elements of the matrix Φ are assumed to be all inside the unit circle, since the number of explosive roots of the original LRE model in (2.1) already equals the number of expectational variables in the model ($m^*(\theta) = 0$). Also, in this case the augmented representation is determinate due to the stability of the appended auxiliary processes. Importantly, as shown for the univariate example in Section 1, the block structure of the proposed methodology guarantees that the autoregressive process, ω_t, does not affect the solution for the endogenous variables, X_t.

Denoting the newly defined vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)^T$ and the newly defined vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)^T$, the system in (2.1) and (2.4) can be written as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t, \tag{2.5}$$

where

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1(\theta) & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n(\theta) & \Pi_f(\theta) \\ 0 & -\mathbf{I} \end{bmatrix},$$

and without loss of generality the matrix Π in (2.1) is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$, where the matrices Π_n and Π_f are respectively of dimension $k \times (p - m)$ and $k \times m$.

Section 2.1 and 2.2 show that the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) and Farmer et al. (2015). In order to simplify the exposition, when analyzing the case of indeterminacy we assume, without loss of generality, $m^*(\theta) = m$. As it will become clear, the case of $m^*(\theta) < m$ is a special case of what we present below.

2.1 Equivalence under determinacy

This section considers the case in which the original LRE is determinate, and shows the equivalence of the solution obtained using the proposed augmented representation with the one from the standard solution algorithm described in Sims (2001b).

2.1.1 Canonical solution

Consider the LRE model in (2.1) and reported below as (2.6)

$$\underset{k\times k}{\Gamma_0}\underset{k\times 1}{X_t} = \underset{k\times k}{\Gamma_1}\underset{k\times 1}{X_{t-1}} + \underset{k\times}{\Psi}\underset{\times 1}{\varepsilon_t} + \underset{k\times p}{\Pi}\underset{p\times 1}{\eta_t}\,. \tag{2.6}$$

The method described in Sims (2001b) delivers a solution, if it exists, by following four steps. First, Sims (2001b) shows how to write the model in the form

$$SZ^T X_t = TZ^T X_{t-1} + Q\Psi\varepsilon_t + Q\Pi\eta_t, \tag{2.7}$$

where $\Gamma_0 = Q^T SZ^T$ and $\Gamma_1 = Q^T TZ^T$ is the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$, and the $k \times k$ matrices Q and Z are orthonormal, upper triangular and possibly complex. Also, the diagonal elements of S and T contain the generalized eigenvalues of $\{\Gamma_0, \Gamma_1\}$.

Second, since the QZ decomposition is not unique, Sims' algorithm chooses a decomposition that orders the equations so that the absolute values of the ratios of the generalized eigenvalues are placed in an increasing order, that is

$$|t_{jj}| \,/\, |s_{jj}| \geq |t_{ii}| \,/\, |s_{ii}| \quad \text{for } j > i.$$

The algorithm then partitions the matrices S, T, Q and Z as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \qquad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \qquad Z' = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \tag{2.8}$$

where the first block corresponds to the system of equations for which $|t_{jj}| \,/\, |s_{jj}| < 1$ and the second block groups the equations which are characterized by explosive roots, $|t_{jj}| \,/\, |s_{jj}| > 1$.

The third step imposes conditions on the second, explosive block to guarantee the existence of at least one bounded solution. Defining the transformed variables

$$\xi_t \equiv Z^T X_t = \begin{bmatrix} \underset{(k-n)\times 1}{\xi_{1,t}} \\ \underset{n\times 1}{\xi_{2,t}} \end{bmatrix}, \tag{2.9}$$

where n is the number of explosive roots, and the transformed parameters

$$\widetilde{\Psi} \equiv Q^T \Psi, \quad \text{and} \quad \widetilde{\Pi} \equiv Q^T \Pi, \tag{2.10}$$

the second block can be written as

$$\xi_{2,t} = S_{22}^{-1} T_{22} \xi_{2,t-1} + S_{22}^{-1} (\widetilde{\Psi}_2 \varepsilon_t + \widetilde{\Pi}_2 \eta_t). \tag{2.11}$$

Since this system of equations contains the explosive roots of the system, then a bounded solution, if it exists, will set

$$\underset{n\times 1}{\xi_{2,0}} = 0 \tag{2.12}$$

$$\underset{n\times\ell}{\tilde{\Psi}_2}\underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{\tilde{\Pi}_2}\underset{p\times 1}{\eta_t} = 0, \tag{2.13}$$

where n also denotes the number of equations in (2.13).[14]

A necessary condition for the existence of a solution requires that the number of unstable roots (n) equals the number of expectational variables (p). In this section, we are considering the solution under determinacy and this guarantees that there are no degrees of indeterminacy $m^*(\theta) = 0$. The sufficient condition then requires that the columns of the matrix $\tilde{\Pi}_2$ are linearly independent so that there is at least one bounded solution. In that case, the matrix $\tilde{\Pi}_2$ is a square, non-singular matrix and equation (2.13) imposes linear restrictions on the forecast errors, η_t, as a function of the fundamental shocks, ε_t,

$$\eta_t = -\tilde{\Pi}_2^{-1}\tilde{\Psi}_2\varepsilon_t. \tag{2.14}$$

The fourth and last step finds the solution for the endogenous variables, X_t, by combining the restrictions in (2.12) and (2.14) with the system of stable equations in the first block,

$$\begin{aligned}\xi_{1,t} &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1\varepsilon_t + \widetilde{\Pi}_1\eta_t) \\ &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}\left(\widetilde{\Psi}_1 - \widetilde{\Pi}_1\tilde{\Pi}_2^{-1}\tilde{\Psi}_2\right)\varepsilon_t \end{aligned} \tag{2.15}$$

Using the algorithm by Sims (2001b), the solution of the LRE model in (2.6) is described by equations (2.12), (2.14), and (2.15).

[14]Note that the eigenvalues of $S_{22}^{-1}T_{22}$ are all greater than one in absolute value.

2.1.2 Augmented representation

We now consider the methodology proposed in this paper, and we augment the LRE model in (2.6) with the following system of m equations

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}, \qquad \Phi \equiv \begin{bmatrix} \frac{1}{\alpha_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{1}{\alpha_m} \end{bmatrix}$$

where Φ is a $m \times m$ diagonal matrix. Since the original model in (2.6) is determinate, then we assume that all the diagonal elements α_i belong to the interval $[1, 2]$. Therefore, we are appending a system of stable equations, and we show that the solution for the endogenous variables, X_t, is equivalent to the one found in Section 2.1.1. Defining the augmented vector of endogenous variables, $\hat{X}_t \equiv (X_t, \omega_t)^T$ and the augmented vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)^T$, the representation that we propose takes the form

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t, \tag{2.16}$$

where

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \quad \hat{\Psi} \equiv \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n & \Pi_f \\ 0 & -\mathbf{I} \end{bmatrix},$$

and without loss of generality the matrix Π is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$, where the matrices Π_n and Π_f are respectively of dimension $k \times (p - m)$ and $k \times m$.

We can find a solution to the augmented representation in (2.16) by using Sims' algorithm. Similarly to the previous section, we follow the four steps which describe the algorithm. First, the solution algorithm performs the QZ decomposition of the matrices $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$ and the augmented representation takes the form

$$\hat{S}\hat{Z}^T\hat{X}_t = \hat{T}\hat{Z}^T\hat{X}_{t-1} + \hat{Q}\hat{\Psi}\hat{\varepsilon}_t + \hat{Q}\hat{\Pi}\eta_t, \tag{2.17}$$

where $\hat{\Gamma}_0 = \hat{Q}^T\hat{S}\hat{Z}^T$ and $\hat{\Gamma}_1 = \hat{Q}^T\hat{T}\hat{Z}^T$ is the QZ decomposition of $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$, and

$$\hat{S} = \begin{bmatrix} S_{11} & \mathbf{0} & S_{12} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_{22} \end{bmatrix}, \quad \hat{T} = \begin{bmatrix} T_{11} & \mathbf{0} & T_{12} \\ \mathbf{0} & \Phi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & T_{22} \end{bmatrix}, \quad \hat{Z}^T = \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ Z_2 & \mathbf{0} \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} Q_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ Q_2 & \mathbf{0} \end{bmatrix}.$$

Importantly, note that the inner matrices of $\{\hat{S}, \hat{T}, \hat{Z}^T, \hat{Q}\}$ are the same as those which define the matrices $\{S, T, Z^T, Q\}$ in (2.8) found in the previous section.

Second, the algorithm chooses a QZ decomposition which groups the equations in a stable and an explosive block. Since this section assumes that the original model is determinate and that the diagonal elements of the matrix Φ are within the unit circle, the explosive block corresponds to the third system of equations in (2.17) which is characterized by explosive roots. Recalling the definition of the matrices $\hat{\Psi}$ and $\hat{\Pi}$, the system of equations in the third block is

$$\xi_{2,t} = S_{22}^{-1}T_{22}\xi_{2,t-1} + S_{22}^{-1}(\widetilde{\Psi}_2\varepsilon_t + \widetilde{\Pi}_2\eta_t). \tag{2.18}$$

The third step imposes conditions to guarantee the existence of a bounded solution. However, the explosive block in (2.18) is identical to the system of equations found in (2.11). Therefore, the algorithm imposes the same restrictions to guarantee the existence of a bounded solution, that is

$$\xi_{2,0} = 0 \tag{2.19}$$

and as found earlier

$$\eta_t = -\widetilde{\Pi}_2^{-1}\widetilde{\Psi}_2\varepsilon_t. \tag{2.20}$$

Finally, the last step combines these restrictions with the system of equations in the stable block which corresponds to the first and second systems of equations in (2.17),

$$\xi_{1,t} = S_{11}^{-1} T_{11} \xi_{1,t-1} + S_{11}^{-1} \left(\widetilde{\Psi}_1 - \widetilde{\Pi}_1 \tilde{\Pi}_2^{-1} \tilde{\Psi}_2 \right) \varepsilon_t, \tag{2.21}$$

$$\omega_t = \Phi \omega_{t-1} + \nu_t - \eta_{f,t}. \tag{2.22}$$

The solution in (2.19)∼(2.22) obtained for the augmented representation of the LRE model delivers the same solution for the endogenous variables of interest, X_t, found in the previous section and defined in equations (2.12), (2.14), and (2.15).

Two remarks should be made when comparing the two solutions. First, as shown in (2.20), the forecast errors are only a function of the exogenous shocks ε_t, and *not* of the sunspot shocks v_t. It is therefore clear that the endogenous variables, X_t, of the original LRE model in (2.6) do not respond to sunspot shocks either, as expected under determinacy. Second, (2.21) and (2.22) indicate that under determinacy the appended system of equations constitutes a separate block, which does not affect the dynamics of the endogenous variables, X_t, and therefore the likelihood function associated with X_t.

2.2 Equivalence under indeterminacy

This section shows the equivalence of the solutions obtained for a LRE model under indeterminacy using the proposed augmented representation and the methodology of Lubik and Schorfheide (2003, 2004).

2.2.1 Lubik and Schorfheide (2003)

As in Section 2.1, we consider the LRE model in (2.6) and reported below as (2.23)

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi \varepsilon_t + \Pi \eta_t. \tag{2.23}$$

In this section we assume that the model is indeterminate, and we present the method used by Lubik and Schorfheide (2003). The authors implement the first two steps of the algorithm by Sims (2001b) and described in Section 2.1.1.[15] They proceed by first applying the QZ decomposition to the LRE model in (2.23) and then ordering the resulting system of equations in a stable and an explosive block as defined in (2.7) and (2.8). However, their approach differs in the third step when the algorithm imposes restrictions to guarantee the existence of a bounded solution. In particular, the restrictions in (2.12) and (2.13) reported below as (2.24) and (2.25) require that

$$\underset{n\times 1}{\xi_{2,0}} = 0, \tag{2.24}$$

$$\underset{n\times\ell}{\tilde{\Psi}_2}\underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{\tilde{\Pi}_2}\underset{p\times 1}{\eta_t} = 0. \tag{2.25}$$

Nevertheless, it is clear that the system of equation in (2.25) is indeterminate since the number of forecast errors exceeds the number of explosive roots ($p > n$). Equivalently, there are less equations (n) than the number of variables to solve for (p). To characterize the full set of solutions to equation (2.25), Lubik and Schorfheide (2003) decompose the matrix $\tilde{\Pi}_2$ using the following singular value decomposition

$$\underset{n\times p}{\tilde{\Pi}_2} \equiv \underset{n\times n}{U}\begin{bmatrix} \underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}} \end{bmatrix}\underset{p\times p}{V^T},$$

where m represents the degrees of indeterminacy. Given the partition $\underset{p\times p}{V} \equiv \begin{bmatrix} \underset{p\times n}{V_1} & \underset{p\times m}{V_2} \end{bmatrix}$, equation (2.25) can be written as

$$\underset{n\times n}{D_{11}^{-1}}\underset{n\times n}{U^T}\underset{n\times\ell}{\tilde{\Psi}_2}\underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{V_1^T}\underset{p\times 1}{\eta_t} = 0. \tag{2.26}$$

[15] It is relevant to mention that in this section the matrices obtained from the QZ decomposition and the ordering of the equations into a stable and an explosive block differ from those in Section 2.1 both in terms of the elements constituting them and their dimensions. However, we opted to use the same notation for simplicity.

Since the system is indeterminate, Lubik and Schorfheide append additional m equations,

$$\underset{m\times\ell}{\widetilde{M}}\underset{\ell\times 1}{\varepsilon_t} + \underset{m\times m}{M_\zeta}\underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2^T}\underset{p\times 1}{\eta_t}. \tag{2.27}$$

The $m \times 1$ vector ζ_t is a set of sunspot shocks that is assumed to have mean zero, covariance matrix $\Omega_{\zeta\zeta}$ and to be uncorrelated with the fundamental shocks, ε_t, that is

$$E[\zeta_t] = 0, \quad E\left[\zeta_t \varepsilon_t^T\right] = 0, \quad E\left[\zeta_t \zeta_t^T\right] = \Omega_{\zeta\zeta}.$$

The matrix $\widetilde{M}$ captures the correlation of the forecast errors, η_t, with fundamentals, ε_t, and Lubik and Schorfheide (2003) choose the normalization $M_\zeta = I_m$. Appending the system of equations in (2.27) to the equations in (2.26), the expectational errors can be written as a function of the fundamental shocks, ε_t, and the sunspot shocks, ζ_t,

$$\underset{p\times 1}{\eta_t} = \left(-\underset{p\times n}{V_1}\underset{n\times n}{D_{11}^{-1}}\underset{n\times n}{U_1^T}\underset{n\times\ell}{\tilde{\Psi}_2} + \underset{p\times m}{V_2}\underset{m\times\ell}{\widetilde{M}}\right)\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2}\underset{m\times 1}{\zeta_t}.$$

More compactly,

$$\underset{p\times 1}{\eta_t} = \left(\underset{p\times n}{V_1}\underset{n\times\ell}{N} + \underset{p\times m}{V_2}\underset{m\times\ell}{\widetilde{M}}\right)\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2}\underset{m\times 1}{\zeta_t}, \tag{2.28}$$

where

$$\underset{n\times\ell}{N} \equiv -\underset{n\times n}{D_{11}^{-1}}\underset{n\times n}{U_1^T}\underset{n\times\ell}{\tilde{\Psi}_2}.$$

is a function of the parameters of the model.

Given the earlier restriction in (2.24) and (2.28), the fourth step in the algorithm combines these equations with the system of stable equations in the first block as in Section 2.1.1,

$$\begin{aligned}\xi_{1,t} &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1\varepsilon_t + \widetilde{\Pi}_1\eta_t) \\ &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}\left(\widetilde{\Psi}_1 + \widetilde{\Pi}_1 V_1 N + \widetilde{\Pi}_1 V_2 \widetilde{M}\right)\varepsilon_t + S_{11}^{-1}\left(\widetilde{\Pi}_1 V_2\right)\zeta_t. \end{aligned} \tag{2.29}$$

Using the method in Lubik and Schorfheide (2003), the solution for the original LRE model under indeterminacy is described by (2.24), (2.28) and (2.29).

2.2.2 Augmented representation

We now consider the augmented representation as in (2.16) and reported below as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t, \tag{2.30}$$

where $\hat{X}_t \equiv (X_t, \omega_t)^T$, $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)^T$ and

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n & \Pi_f \\ 0 & -\mathbf{I} \end{bmatrix}. \tag{2.31}$$

where the matrix Π is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$ without loss of generality.

The novelty of our approach is that, given our representation, we can easily obtain the solution by using Sims' algorithm even when the original LRE is assumed to be indeterminate. It is enough to assume that the auxiliary processes ω_t are characterized by explosive roots, or equivalently that the diagonal elements of the matrix Φ are outside the unit circle. This guarantees that the Blanchard-Kahn condition for the augmented representation is satisfied and, given the analytic form that we propose for the auxiliary processes, we show that the solution for the endogenous variables of interest, X_t, is equivalent to the method of Lubik and Schorfheide (2003).

To show this result, we simply apply the four steps of the algorithm described in Sims (2001b) to the proposed augmented representation. First, the QZ decomposition of (2.30) takes the form

$$\hat{S}\hat{Z}^T \hat{X}_t = \hat{T}\hat{Z}^T \hat{X}_{t-1} + \hat{Q}\hat{\Psi}\hat{\varepsilon}_t + \hat{Q}\hat{\Pi}\eta_t, \tag{2.32}$$

where $\hat{\Gamma}_0 = \hat{Q}^T \hat{S} \hat{Z}^T$ and $\hat{\Gamma}_1 = \hat{Q}^T \hat{T} \hat{Z}^T$ is the QZ decomposition[16] of $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$ and

$$\hat{S} = \begin{bmatrix} S_{11} & S_{12} & \mathbf{0} \\ \mathbf{0} & S_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \hat{T} = \begin{bmatrix} T_{11} & T_{12} & \mathbf{0} \\ \mathbf{0} & T_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Phi \end{bmatrix}, \quad \hat{Z}^T = \begin{bmatrix} Z_1 & \mathbf{0} \\ Z_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} Q_1 & \mathbf{0} \\ Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{2.33}$$

Second, the QZ decomposition chosen by the algorithm groups the explosive dynamics of the model in the second and third system of equations in (2.32), which are reported below as (2.34)

$$\begin{bmatrix} S_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \xi_2 \\ \omega_t \end{bmatrix} = \begin{bmatrix} T_{22} & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix} \begin{bmatrix} \xi_{2,t-1} \\ \omega_{t-1} \end{bmatrix} + \begin{bmatrix} Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left(\hat{\Psi} \hat{\varepsilon}_t + \hat{\Pi} \eta_t \right), \tag{2.34}$$

where $\xi_t \equiv Z^T X_t$.

In the third step, the following restrictions are imposed,

$$\underset{n\times 1}{\xi_{2,0}} = 0, \tag{2.35}$$

$$\underset{m\times 1}{\omega_0} = 0, \tag{2.36}$$

$$\begin{bmatrix} Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left(\hat{\Psi} \hat{\varepsilon}_t + \hat{\Pi} \eta_t \right) = 0. \tag{2.37}$$

Recalling the definition of $\hat{\Psi}$ and $\hat{\Pi}$ and that we define $\widetilde{\Psi} \equiv Q^T \Psi$, and $\widetilde{\Pi} \equiv Q^T \Pi$, then equation (2.37) can be written as

$$\underbrace{\begin{bmatrix} \widetilde{\Psi}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}}_{p\times(\ell+m)} \underset{(\ell+m)\times 1}{\hat{\varepsilon}_t} + \underbrace{\begin{bmatrix} \widetilde{\Pi}_{n,2} & \widetilde{\Pi}_{f,2} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}}_{p\times p} \underset{p\times 1}{\eta_t} = 0. \tag{2.38}$$

[16]Note that the inner matrices of $\{\hat{S}, \hat{T}, \hat{Z}^T, \hat{Q}\}$ are the same as those which define the matrices $\{S, T, Z^T, Q\}$ found from the QZ decomposition using the methodology of Lubik and Schorfheide (2003).

Equation (2.38) shows transparently how the explosive auxiliary process that we append help to solve the original LRE model under indeterminacy. The system of equations in (2.38) is determinate as the number of equations defined by the explosive roots of the system equals the number of expectational errors of the model. This guarantees that the necessary condition for the existence of a bounded solution for the augmented representation is satisfied. Assuming that the columns of the matrix associated with the vector of non-fundamental shocks, η_t, are linearly independent, we can therefore impose linear restrictions on the forecast errors as a function of the augmented vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)^T$,

$$\eta_t = -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2 & \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \hat{\varepsilon}_t. \tag{2.39}$$

More compactly,

$$\eta_t = C_1\varepsilon_t + C_2\nu_t, \tag{2.40}$$

where $C_1 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2 \\ \mathbf{0} \end{bmatrix}$ and $C_2 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2} \\ -\mathbf{I} \end{bmatrix}$ are a function of the structural parameters of the model.

The last step of Sims' algorithm combines the restrictions in (2.35), (2.36) and (2.40) with the stationary block derived from the QZ decomposition in (2.32),

$$\begin{aligned} \xi_{1,t} &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1\varepsilon_t + \widetilde{\Pi}_1\eta_t) \\ &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}\left(\widetilde{\Psi}_1 + \widetilde{\Pi}_1 C_1\right)\varepsilon_t + S_{11}^{-1}\left(\widetilde{\Pi}_1 C_2\right)v_t. \end{aligned} \tag{2.41}$$

2.2.3 Indeterminate equilibria and equivalent characterizations

The indeterminate equilibria found using the methodology of Lubik and Schorfheide (2003) are parametrized by two sets of parameters. The first set is defined by $\theta_1 \in \Theta_1$, where $\theta_1 \equiv vec(\Gamma_0, \Gamma_1, \Psi, \Omega_{\varepsilon\varepsilon})^T$ is a vector of structural parameters of the model as well as the covariance matrix of the exogenous shocks. The second set corresponds to $\theta_2 \in \Theta_2$, where

$\theta_2 \equiv vec\left(\Omega_{\zeta\zeta}, \widetilde{M}\right)^T$ is a parameter vector related to the additional equations introduced in (2.27) and reported below as (2.42),

$$\underset{m\times\ell}{\widetilde{M}}\underset{\ell\times 1}{\varepsilon_t} + \underset{m\times m}{M_\zeta}\underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2^T}\underset{p\times 1}{\eta_t}. \tag{2.42}$$

Given the normalization $M_\zeta = I$ chosen by Lubik and Schorfheide (2004), equation (2.42) introduces $m \times (m+1)/2$ parameters associated with the covariance matrix of the sunspot shocks, $\Omega_{\zeta\zeta}$, and additional $m \times \ell$ parameters of the matrix $\widetilde{M}$ that is related to the covariances between η_t and ε_t. In Appendix A, we show how the normalization chosen in Lubik and Schorfheide (2004) maps into the methodology we propose.

The characterization of a Lubik-Schorfheide equilibrium is a vector $\theta^{LS} \in \Theta^{LS}$, where Θ^{LS} is defined as

$$\Theta^{LS} \equiv \{\Theta_1, \Theta_2\}. \tag{2.43}$$

Similarly, the full characterization of the solutions under indeterminacy using the proposed augmented representation is parametrized by the set of parameters $\theta_1 \in \Theta_1$ common between the two methodologies, and the set of additional parameters $\theta_3 \in \Theta_3$, where $\theta_3 \equiv vec(\Omega_{\nu\nu}, \Omega_{\nu\varepsilon})^T$. Using our approach, we also introduce $m \times (m+1)/2$ parameters associated with the covariance matrix of the sunspot shocks, Ω_{vv}, and $m \times \ell$ parameters of the covariances, $\Omega_{v\varepsilon}$, between the sunspot shock v_t and the exogenous shocks ε_t. A Bianchi-Nicolò equilibrium is characterized by a parameter vector $\theta^{BN} \in \Theta^{BN}$, where Θ^{BN} is defined as

$$\Theta^{BN} \equiv \{\Theta_1, \Theta_3\}. \tag{2.44}$$

The following theorem establishes the equivalence between the characterizations of indeterminate equilibria obtained by using the methodology in Lubik and Schorfheide (2003) and the proposed augmented representation.

Theorem 3. *Let θ^{LS} and θ^{BN} be two alternative parametrizations of an indeterminate equi-*

librium of the model

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi \varepsilon_t + \Pi \eta_t.$$

For every BN equilibrium, parametrized by θ^{BN}*, there is a unique matrix* $\tilde{M}$ *and a unique matrix* $\Omega_{\zeta\zeta}$ *such that* $\theta_2 = vec(\Omega_{\zeta\zeta}, \tilde{M})^T$*, and* $\{\theta_1, \theta_2\} \in \Theta^{LS}$ *defines an equivalent LS equilibrium. Conversely, for every LS equilibrium, parametrized by* θ^{LS}*, there is a unique matrix* Ω_{vv} *and a unique covariance matrix* $\Omega_{v\varepsilon}$ *such that* $\theta_3 = vec(\Omega_{vv}, \Omega v\varepsilon)^T$*, and* $\{\theta_1, \theta_3\} \in \Theta^{BN}$ *defines an equivalent BN equilibrium.*

Proof. See Appendix B. □

In the paper Farmer et al. (2015), the authors also show that their characterization of indeterminate equilibria is equivalent to Lubik and Schorfheide (2003). Therefore, the following corollary holds.

Corollary 2. *Given a parametrization* θ^{BN} *of a Bianchi-Nicolò indeterminate equilibrium, there exists a unique mapping into the parametrization of an indeterminate equilibrium for Farmer et al. (2015), and vice-versa.*

Moreover, the following two considerations support Corollary 3 below, which describes a relevant result on the likelihood function of the augmented representation. First, as emphasized in this section, the reduced form of the augmented representation has a block structure which ensures that solution for endogenous variables in X_t is not a function of the autoregressive processes, ω_t. Second, note that the appended autoregressive processes, ω_t, have no economic interpretation and therefore have no relation with the observable variables used in a measurement equation. These two considerations imply that the parameters of the matrix Φ introduced with the augmented representation are not identified (within a certain region). Corollary 3 then follows.[17]

[17]Notice that Corollary 3 holds when the augmented representation has a unique solution. This happens in two cases. First, values of the structural parameters θ which guarantee determinacy in the original LRE

Corollary 3. *The likelihood function associated with the newly defined vector of endogenous variables, $\hat{X}_t$, does not depend on the additional parameters included in the augmented representation, Φ, and is equivalent to the likelihood function associated with the endogenous variables, X_t.*

While Section 2.1 shows that the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, Theorem 3 proves that under indeterminacy the solutions of our methodology are equivalent to those obtained using Lubik and Schorfheide (2003, 2004) and Farmer et al. (2015). This theoretical result is crucial for the estimation exercises conducted in Section 4. The augmented representation guarantees that the Metropolis-Hastings algorithm explores the entire domain of the parameter space.

3 Analytic example

This section considers the canonical NK model to provide an analytic example of the theoretical result presented in Section 2. Let

$$x_t = E_t(x_{t+1}) - \tau(R_t - E_t(x_{t+1})) \tag{3.1}$$

$$\pi_t = \beta E_{t-1}(\pi_{t+1}) + \kappa x_t \tag{3.2}$$

$$R_t = \psi \pi_t + \varepsilon_{R,t} \tag{3.3}$$

$$\eta_{1,t} = x_t - E_{t-1}(x_t) \tag{3.4}$$

$$\eta_{2,t} = \pi_t - E_{t-1}(\pi_t) \tag{3.5}$$

where equations (3.1)$\sim$(3.3) represent the dynamic IS curve, the NK Phillips curve and a monetary policy reaction function, respectively. The variable x_t represents log deviations

model should be combined with values for α_i in the matrix Φ whose absolute value lies within the unit circle. Second, values of the structural parameters θ for which the original model is indeterminate should be combined with (absolute) values of α_i outside the unit circle.

of GDP from a trend path and π_t and R_t are log deviations from the steady state level of inflation and the nominal interest rate. The one-step ahead forecast errors for the deviations of output from its trend and of inflation from its steady state are defined in (3.4) and (3.5). This model can be expressed in matrix form as

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi \varepsilon_t + \Pi \eta_t, \tag{3.6}$$

where $X_t = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}))^T$, $\varepsilon_t = (\varepsilon_{R,t})$ and $\eta_t = (\eta_{1,t}, \eta_{2,t})^T$.

It is well known that the region of determinacy is associated with an active response of the monetary authority to changes in inflation, a condition satisfied when $|\psi| > 1$. Alternatively, the equilibrium is indeterminate when the monetary policy is "passive", that is $0 < |\psi| \leq 1$. In the latter case, there is one degree of indeterminacy ($m = 1$) since there are two forecast errors while the system is characterized by only one unstable root.[18] Given that $m = 1$, the proposed methodology consists in appending to the original LRE model in (3.6) the following equation

$$\omega_t = \frac{1}{\alpha}\omega_{t-1} + \nu_t - \eta_{2,t}. \tag{3.7}$$

To provide the intuition, consider $\alpha \equiv |\psi|$. When the monetary authority is "passive," indeterminacy arises and the Blanchard-Kahn condition for the original LRE model is not satisfied. Our representation augments the system in (3.6) with the explosive autoregressive process in (3.7). Note that under indeterminacy $0 < |\psi| \leq 1$, which implies $1/\alpha > 1$. The augmented representation not only mechanically satisfies the Blanchard-Kahn condition, but, as proven in Theorem 3, it describes all the set of equilibria which would be equivalently obtained using the methodology of Lubik and Schorfheide (2003, 2004) or Farmer et al. (2015). Alternatively, when the monetary policy adopts an "active" stance, the original

[18] As shown in Appendix C, one of the roots of the system is always outside the unit circle. This implies that the maximum degree of indeterminacy is $m = 1$, and that we append only one auxiliary autoregressive process. However, it is not always possible to derive this analytical results. Under those circumstances, the number of auxiliary processes to append to the original LRE model equals the number of expectational variable. The latter indeed corresponds to the maximum degree of indeterminacy of the original model.

system is determinate and the auxiliary autoregressive process is stationary (i.e. $0 < 1/\alpha < 1$), thus satisfying the Blanchard-Kahn condition under determinacy. Importantly, as shown both in this example and more generally in Section 2, the block structure of the augmented representation ensures that the endogenous variables contained in the vector X_t are not a function of the process ω_t for both regions of the parameter space.

We now show that the equivalence results in Section 2 hold for the NK model described by (3.1) $\sim$ (3.5). As described in the previous section, the proposed methodology defines a new vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)^T = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}), \omega_t)^T$ and a newly defined vector of exogenous shocks as $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)^T = (\varepsilon_{R,t}, \nu_t)^T$. The system in (3.6) and (3.7) can then be written as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t. \tag{3.8}$$

The representation in (3.8) always delivers a solution of the following form under both determinacy and indeterminacy,

$$\hat{X}_t = \hat{T}\hat{X}_{t-1} + \hat{R}\hat{\varepsilon}_t. \tag{3.9}$$

In what follows, we show the equivalence of our solutions under determinacy with the one from Sims (2001b), and under indeterminacy with the solution proposed by Farmer et al. (2015).

3.1 Determinacy

This section clarifies the details for the equivalence of the solutions which are obtained in the *determinacy* region of the parameter space when using the following two representations:

a) The matrix representation of the model in (3.6) and reported here as equation (3.10)

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi\varepsilon_t + \Pi\eta_t \tag{3.10}$$

Equivalence of solutions under determinacy	
Sims (2001b)	Bianchi-Nicolò
$E_t(x_{t+1}) = E_t(\pi_{t+1}) = 0$	$E_t(x_{t+1}) = E_t(\pi_{t+1}) = 0$
$\eta_t = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1\\ \kappa\end{bmatrix}\varepsilon_{R,t}$	$\eta_t = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1 & 0\\ \kappa & 0\end{bmatrix}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix}$
$\begin{pmatrix}x_t\\ \pi_t\end{pmatrix} = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1\\ \kappa\end{bmatrix}\varepsilon_{R,t}$	$\begin{pmatrix}x_t\\ \pi_t\end{pmatrix} = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1 & 0\\ \kappa & 0\end{bmatrix}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix}$
-	$\omega_t = \left(\frac{1}{\alpha}\right)\omega_{t-1} + \begin{bmatrix}\frac{\tau\kappa}{1+\kappa\tau\psi} & 1\end{bmatrix}\begin{bmatrix}\varepsilon_{R,t}\\ \nu_t\end{bmatrix}$

Table 3.1: Equivalence of solutions under determinacy. The table compares the solution obtained by using our methodology with Sims (2001).

b) The proposed augmented representation in (3.8) and reported here as equation (3.11)

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t. \tag{3.11}$$

Representations a) and b) deliver the equilibrium conditions reported in Table 3.1,

where $\alpha \equiv |\psi| > 1$.[19] Comparing the obtained solutions, it is clear that they are equivalent. While our augmented representation potentially allows for the sunspot shock to affect the model dynamics, the coefficients which determine its impact on the endogenous variables equal zero. Moreover, the dynamics of the endogenous variables $X_t = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}))'$ are not affected by the autoregressive process ω_t since it constitutes a separate block.

[19] Details on the derivation of the solutions are provided in the Appendix C.

3.2 Indeterminacy

Under indeterminacy, the Blanchard-Kahn condition is not satisfied and to solve the model we use the solution method suggested by Farmer et al. (2015).[20] The solution obtained using the method of Farmer et al. (2015) is equivalent to Lubik and Schorfheide (2003,2004). We use Farmer et al. (2015) solution because easier to compare with our solution. Hence, the solutions that we compare in this section derive from the following two representations:

c) The matrix representation of the LRE model using the methodology of Farmer et al. (2015) when the forecast error for the deviations of inflation from its steady state, $\eta_{2,t}$, is included as newly defined fundamental shock. Given the partition of the matrix Π in (3.6) as $\Pi = [\Pi_n \quad \Pi_f]$, then

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi_f \tilde{\varepsilon}_t + \Pi_n \eta_{1,t} \tag{3.12}$$

where $\tilde{\varepsilon}_t \equiv (\varepsilon_t, \eta_{2,t})^T$ and $\Psi_f \equiv [\Psi \quad \Pi_f]$.

d) The proposed augmented representation, equivalent to the representation b) in Section 3.1

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t.$$

The equilibrium conditions obtained using representations c) and d) are reported in Table 3.2, where

$$\underset{4\times 1}{G} \equiv \begin{pmatrix} -\frac{a_2}{2\kappa} \\ 1 \\ -\frac{a_1 a_2}{4\beta\kappa} \\ \frac{a_1}{2\beta} \end{pmatrix} \qquad \underset{4\times 2}{H} \equiv \begin{pmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \\ 0 & 1 \\ -\frac{\tau a_2}{a_3} & -\frac{a_2(1+\kappa\tau\psi)}{a_3\kappa} \\ \frac{2\kappa\tau}{a_3} & -\frac{2(1+\kappa\tau\psi)}{a_3} \end{pmatrix} \tag{3.13}$$

[20]The derivation of the solutions obtained using the method by Farmer et al. (2015) and the proposed augmented representation are provided in Appendix D.

and $a_1 = (\beta - \phi + \kappa\tau + 1)$, $a_2 = (a_1 - 2)$, $a_3 = (a_1 + 2\phi)$ and $\phi = [(1 + \beta + \kappa\tau)^2 - 4\beta(1 + \kappa\tau\psi)]^{-1/2}$. To understand the equivalence result, it is useful to compare the linear restrictions imposed on the vector of forecast errors using the augmented representation. In particular, note that our methodology imposes the restriction $\eta_{2,t} = \nu_t$. Thus, the solution to the augmented representation sets restrictions on the forecast error, $\eta_{2,t}$, (which has been redefined as fundamental using the methodology of Farmer et al. (2015)) such that it corresponds to the sunspot shock, ν_t. Also, to guarantee a bounded solution, restrictions are imposed such that the autoregressive process ω_t equals zero at any time t. Therefore, the solutions for the two alternative representations are equivalent.

Also, two relevant comments can be made. First, under indeterminacy the endogenous variables are also affected by the sunspot shock. Second, comparing the form of the matrices under determinacy in Table 3.1 with those under indeterminacy in Table 3.2, it is evident that the propagation mechanism differs according to which region of the parameter space is considered.

4 Applications

While the previous section provides an analytic example clarifying the equivalence results shown in Section 2, this section highlights the importance of our results for the estimation of LRE models. We consider the three-equation NK model of Lubik and Schorfheide (2004) and we conduct the following exercises. Section 4.1 and Section 4.2 deal with simulated data. In particular, we run two simulations of the model for parameter values which lie in the region of the parameter space associated with determinacy and indeterminacy. Given the two simulations, Section 4.1 assumes that the region of determinacy is *known*. In Section 4.2, we then assume that the region of determinacy is *unknown*. In both cases the MCMC algorithm converges to the correct area of the parameter space. Section 4.3 then provides an example on how to implement our methodology when using real data. We consider the data of Lubik

Equivalence of solutions under indeterminacy	
Farmer et al. (2015)	Bianchi-Nicolò
$E_t(x_{t+1}) = -\frac{a_2}{2\kappa}E_t(\pi_{t+1})$	$E_t(x_{t+1}) = -\frac{a_2}{2\kappa}E_t(\pi_{t+1})$
$\eta_{1,t}=\begin{bmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \end{bmatrix}\begin{bmatrix} \varepsilon_{R,t} \\ \eta_{2,t} \end{bmatrix}$	$\eta_t=\begin{bmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \varepsilon_{R,t} \\ \nu_t \end{bmatrix}$
$\begin{pmatrix} x_t \\ \pi_t \\ E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix}=\underset{4\times1}{G}E_{t-1}(\pi_t)+\underset{4\times2}{H}\begin{bmatrix} \varepsilon_{R,t} \\ \eta_{2,t} \end{bmatrix}$	$\begin{pmatrix} x_t \\ \pi_t \\ E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix}=\underset{4\times1}{G}E_{t-1}(\pi_t)+\underset{4\times2}{H}\begin{bmatrix} \varepsilon_{R,t} \\ \nu_t \end{bmatrix}$
-	$\omega_t= 0$

Table 3.2: Equivalence of solutions under indeterminacy. The table compares the solution obtained by using our methodology with Farmer et al. (2015). This last method, in turn, returns the same solution obtained by applying the methods of Lubik and Schorfheide (2003,2004).

and Schorfheide (2004) for the period prior to the appointment of Chairman Paul Volcker, and we retain the assumption that the researcher does not know the region of determinacy. We show that our method enables to successfully recover the same posterior distributions reported by Lubik and Schorfheide (2004), regardless of the region of the parameter space in which the estimation is initialized. Finally, we run the estimation several times and verify that the results presented below hold across all of them.

We consider the NK model estimated by Lubik and Schorfheide (2004). The model is described by equations (4.1)$\sim$(4.6) and consists of a dynamic IS curve

$$x_t = E_t\left(x_{t+1}\right) - \tau\left(R_t - E_t\left(\pi_{t+1}\right)\right) + g_t, \tag{4.1}$$

a NK Phillips curve

$$\pi_t = \beta E_t\left(\pi_{t+1}\right) + \kappa\left(x_t - z_t\right), \tag{4.2}$$

and a Taylor rule,

$$R_t = \rho_R R_{t-1} + (1 - \rho_R) \left[\psi_1 \pi_t + \psi_2 \left(x_t - z_t\right)\right] + \varepsilon_{R,t}. \tag{4.3}$$

The demand shock, g_t, and the supply shock, z_t, follow univariate AR(1) processes

$$g_t = \rho_g g_{t-1} + \varepsilon_{g,t}, \tag{4.4}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t}, \tag{4.5}$$

where the standard deviations of the fundamental shocks $\varepsilon_{g,t}$, $\varepsilon_{z,t}$ and $\varepsilon_{R,t}$ are denoted by σ_g, σ_z and σ_R, respectively. As in Lubik and Schorfheide (2004), we allow for the correlation between demand and supply shocks, ρ_{gz}, to be nonzero. The rational expectation forecast errors are defined as

$$\eta_{1,t} = x_t - E_{t-1}\left[x_t\right], \qquad \eta_{2,t} = \pi_t - E_{t-1}\left[\pi_t\right]. \tag{4.6}$$

We define the vector of endogenous variables as $X_t \equiv \left(x_t, \pi_t, R_t, E_t\left(x_{t+1}\right), E_t\left(\pi_{t+1}\right), g_t, z_t\right)^T$, the vectors of fundamental shocks and non-fundamental errors,

$$\varepsilon_t = \left(\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}\right)^T, \qquad \eta_t = \left(\eta_{1,t}, \eta_{2,t}\right)^T$$

and the vector of parameters

$$\theta = \left(\psi_1, \psi_2, \rho_R, \beta, \kappa, \tau, \rho_g, \rho_z, \sigma_g, \sigma_z, \sigma_R, \rho_{gz}, \rho_{gR}, \rho_{zR}\right)^T.$$

This leads to the following representation of the model,

$$\Gamma_0(\theta) X_t = \Gamma_1(\theta) X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t. \tag{4.7}$$

The LRE model in (4.7) is determinate when the following analytic condition is satisfied, $|\psi^*| > 1$, where $\psi^* \equiv \psi_1 + \frac{(1-\beta)}{\kappa}\psi_2$. However, when the model is indeterminate, $0 < |\psi^*| \leq 1$, the system is characterized by one degree of indeterminacy ($m = 1$) since there are two expectational variables $\{E_t(x_{t+1}), E_t(\pi_{t+1})\}^T$ and only one root outside the unit circle. The methodology we propose consists in augmenting the representation of the model in (4.7) with the autoregressive process

$$\omega_t = \left(\frac{1}{\alpha}\right)\omega_{t-1} + \nu_t - \eta_{2,t}. \tag{4.8}$$

Hence, we define a new vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)^T$ and a newly defined vector of exogenous shocks as $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)^T = (\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}, \nu_t)^T$. The system in (4.7) and (4.8) can then be written as

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t. \tag{4.9}$$

As in Lubik and Schorfheide (2004), the vector of observables, $\mathbf{y}_t = \{x_{obs,t}, \pi_{obs,t}, R_{obs,t}\}$, consists of

1. $x_{obs,t}$ the percentage deviations of (log) real GDP per capita from an HP-trend;

2. $\pi_{obs,t}$ the annualized percentage change in the Consumer Price Index for all Urban Consumers;

3. $R_{obs,t}$ the annualized percentage average Federal Funds Rate.

The measurement equations are described by

$$\mathbf{y}_t = \begin{bmatrix} 0 \\ \pi^* \\ \pi^* + r^* \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \end{bmatrix} X_t. \tag{4.10}$$

Parameter values for simulations		
Parameter	Determinacy	Indeterminacy
ψ_1	2.1	0.73
ψ_2	0.16	0.16
ρ_R	0.67	0.67
π^*	4.03	4.03
r^*	1.22	1.22
κ	0.86	0.86
τ^{-1}	1.61	1.61
ρ_g	0.77	0.77
ρ_z	0.78	0.78
σ_R	0.22	0.22
σ_g	0.24	0.24
σ_z	1.10	1.10
ρ_{gz}	0.46	0.46
σ_ν	-	0.24
$\rho_{R\nu}$	-	-0.19
$\rho_{g\nu}$	-	0.15
$\rho_{z\nu}$	-	-0.21

Table 4.1: Parameter values used for the simulations.

where π^* and r^* are annualized steady-state inflation and real interest rates expressed in percentages. The discount factor, β is a function of the annualized real interest rate in steady-state r^* (i.e. $\beta = (1 + r^*)^{-1/4}$). We then simulate the model under both determinacy and indeterminacy and Table 4.1 reports the parameter values used for the simulations.[21]

While under determinacy we set $\psi_1 = 2.1$ (thus guaranteeing $|\psi^*| > 1$), for the simulation under indeterminacy we impose $\psi_1 = 0.7$ for which $0 < |\psi^*| < 1$. Also, under indeterminacy we use the values for the standard deviation of the sunspot shock and its correlation with the fundamental shocks reported in Farmer et al. (2015). Finally, Table 4.2 reports the prior distributions used for the estimation exercises in the following sections.

[21] The parameter values are those that we estimate in the following section using the data of Lubik and Schorfheide (2004) for the Pre-Volcker period. These estimates are in line with those that the authors report. Also, for the purpose of this paper changing the values assigned to the parameters which are *not* directly related to the analytic condition defining the region of determinacy is irrelevant.

Prior distribution for model parameters					
Name	Range	Density	Mean	Std. Dev.	90% interval
ψ_1	R^+	*Gamma*	1.1	0.50	[0.43,2.03]
ψ_2	R^+	*Gamma*	0.25	0.15	[0.06,0.54]
ρ_R	$[0,1)$	*Beta*	0.50	0.20	[0.17,0.83]
π^*	R^+	*Gamma*	4.00	2.00	[1.35,7.75]
r^*	R^+	*Gamma*	2.00	1.00	[0.69,3.86]
κ	R^+	*Gamma*	0.50	0.20	[0.22,0.87]
τ^{-1}	R^+	*Gamma*	2.00	0.50	[1.25,2.88]
ρ_g	$[0,1)$	*Beta*	0.70	0.10	[0.52,0.85]
ρ_z	$[0,1)$	*Beta*	0.70	0.10	[0.52,0.85]
σ_R	R^+	*Inverse Gamma*	0.31	0.16	[0.14,0.60]
σ_g	R^+	*Inverse Gamma*	0.38	0.20	[0.17,0.74]
σ_z	R^+	*Inverse Gamma*	1.00	0.52	[0.47,1.95]
ρ_{gz}	[-1,1]	*Uniform*	0.00	0.58	[-0.90,0.90]
σ_ν	R^+	*Uniform*	0.5	0.29	[0.05,0.95]
$\rho_{R\nu}$	[-1,1]	*Uniform*	0.00	0.58	[-0.90,0.90]
$\rho_{g\nu}$	[-1,1]	*Uniform*	0.00	0.58	[-0.90,0.90]
$\rho_{z\nu}$	[-1,1]	*Uniform*	0.00	0.58	[-0.90,0.90]

Table 4.2: Prior distributions for the model parameters.

4.1 Known region of determinacy

In this section, we assume that the region of determinacy is *known*. We show that our augmented representation accommodates with a single framework both the case of determinacy and indeterminacy. This feature of our solution method makes it possible for the optimization algorithm to search over the entire parameter space, therefore increasing the probability of finding the posterior mode. As explained in An and Schorfheide (2007), the posterior mode is a crucial object for Bayesian inference. First, the posterior mode is often used as a point estimate for the parameters of the model. Second, it is often used as a starting point for the Metropolis-Hastings algorithm. Finally, a scaled version of the inverse of the Hessian matrix evaluated at the posterior mode is often used as the covariance matrix for the proposal distribution in the Metropolis-Hastings algorithm.

Since we assume that the region of determinacy $|\psi^*| > 1$ is known, we set $\alpha \equiv |\psi^*|$. This assumption implies that when the model is determinate, the autoregressive process is stable

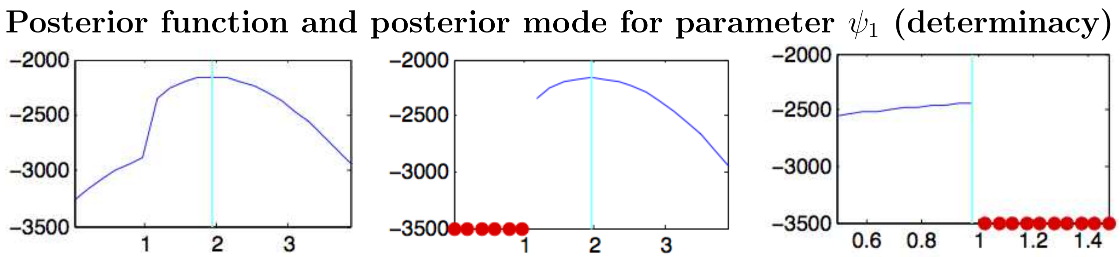

Figure 4.1: Posterior function and posterior mode for the parameter ψ_1 for the augmented representation (panel (a)), the representation under determinacy (panel (b)) and under indeterminacy (panel (c)).

and the solution is equivalent to the solution of the original model (4.7). On the other hand, when the model is indeterminate (i.e. $0 < |\psi^*| \leq 1$), the autoregressive process is unstable, satisfying the Blanchard-Kahn condition. The assumption $\alpha \equiv |\psi^*|$ enables to search for the posterior mode over the *entire* parameter space.

First, we consider the simulation of the model under determinacy, and we compute the posterior mode of the model parameters using three different representations of the Lubik and Schorfheide (2004) model. We consider the augmented representation proposed in this paper, the representation of the model under determinacy using Sims' (2001) algorithm, and the representation of the model under indeterminacy using the methodology of Farmer et al. (2015).[22] Figure 4.1 reports the posterior mode (vertical line) and how the posterior varies to changes in the parameter ψ_1, while keeping the other structural parameters at their posterior mode estimates. While panel (a) considers the augmented representation, panel (b) and (c) report the plots for the representations under model determinacy and indeterminacy, respectively.

The red dots on the horizontal axis in panel b) and c) indicate parameter values for which, given the chosen model representation, the model could not be solved due to a violation of the Blanchard-Kahn condition. While in panel b) the model violates these conditions for values of the parameter ψ_1 smaller than 1, panel c) shows that the representation of Farmer

[22] As in Section 3.2, we apply the methodology of Farmer et al. (2015) by redefining the forecast error for inflation, $\eta_{2,t}$, as fundamental shock, that is $\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi_f \tilde{\varepsilon}_t + \Pi_n \eta_{1,t}$, where $\tilde{\varepsilon}_t = (\varepsilon_{R,t}, \eta_{2,t})'$ and $\Pi = [\Pi_n \quad \Pi_f]$.

et al. (2015) does not allow to solve the model for values of ψ_1 greater than 1.[23] Figure 4.1 highlights that the augmented representation guarantees that the optimization algorithm explores the entire domain of the parameter space. Note that as pointed out by Lubik and Schorfheide (2004) the posterior presents a discrete jump between the two parameter regions. This makes it unlikely that once the correct region of the parameter space is reached, the estimation algorithm will leave such region. We will further elaborate on this point in Section 6.

Similarly, when using the Metropolis-Hastings algorithm, candidate parameter values can be drawn from both the determinacy and the indeterminacy region. However, once the algorithm converges to the "correct" area of the parameter space, it is unlikely to leave it. This is reflected in Table 4.3, that reports the mean and the 90% probability interval of the posterior distributions.[24] The posterior estimates indicate that the true parameter values are recovered under the augmented representation. All the parameter values used to simulate the model fall within the 90% probability intervals of the posterior distributions.

We perform the same estimation exercise using the simulation of the model under indeterminacy. Figure 4.2 plots how the posterior varies with ψ_1 while the other parameters are constant at their posterior mode estimates. As before, the vertical line reports the corresponding posterior mode. Figure 4.2 provides similar evidence as in Figure 4.1. Panel (a), panel (b) and panel (c) refer to the augmented representation and the representation under determinacy and under indeterminacy, respectively.

Contrary to the alternative representations, the proposed augmented representation ensures to run the optimization routine to compute the posterior mode over the entire parameter space. Reasonably, the shape of the maximized functions in panel (a) of Figure 4.2

[23] The violation of the Blanchard-Kahn conditions for values of ψ_1 close to 1 results from the values chosen for the simulation. Indeed, the term $\frac{(1-\beta)}{\kappa}\psi_2 \approx 0$, thus implying that the region of determinacy is approximated by the following condition $\psi^* \approx \psi_1 > 1$.

[24] Since the posterior estimates satisfy the analytic condition for determinacy, the endogenous variables, X_t, are not a function of the sunspot shock and we therefore do not report the estimates of the standard error of the sunspot shock and its correlation with the fundamental shocks.

Posterior estimates, simulation under determinacy			
	True values	Posterior estimates	
		Mean	90% probability interval
ψ_1	2.1	1.90	[1.59,2.22]
ψ_2	0.16	0.34	[0.03,0.62]
ρ_R	0.67	0.67	[0.64,0.70]
π^*	4.03	4.17	[3.95,4.40]
r^*	1.22	1.39	[1.13,1.65]
κ	0.86	0.71	[0.44,0.98]
τ^{-1}	1.61	1.66	[1.22,2.10]
ρ_g	0.77	0.75	[0.70,0.79]
ρ_z	0.78	0.77	[0.73,0.82]
σ_R	0.22	0.21	[0.20,0.22]
σ_g	0.24	0.26	[0.22,0.30]
σ_z	1.10	1.07	[0.98,1.15]
ρ_{gz}	0.46	0.35	[0.15,0.58]

Table 4.3: Posterior distributions obtained by estimating the model using the simulation under determinacy.

Posterior function and posterior mode for parameter ψ_1 (indeterminacy)

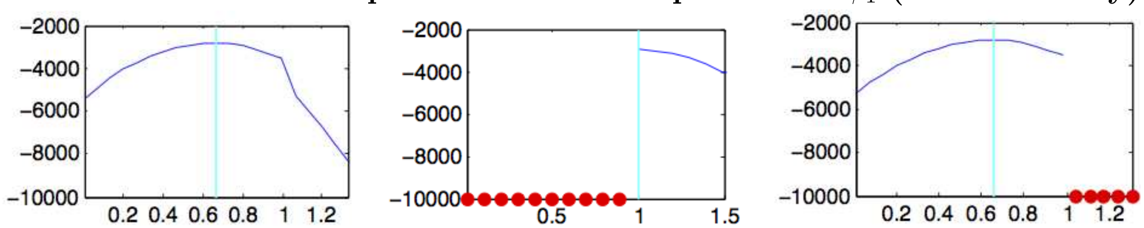

Figure 4.2: Posterior function and posterior mode for the parameter ψ_1 for the augmented representation (panel (a)), the representation under determinacy (panel (b)) and under indeterminacy (panel (c)).

mirrors the plot of panel (a) in Figure 4.1, with the difference that now the peak for the posterior occurs in the indeterminacy region. Table 4.4 reports the mean and 90% probability of the posterior distribution. Also in this case, we recover the true parameter values by using the proposed augmented representation. Since the posterior estimates indicate that the model is characterized by indeterminacy, we report the standard error of the sunspot shock, σ_ν, and its covariance with the fundamental shocks (i.e. $\rho_{\nu R}, \rho_{\nu g}, \rho_{\nu z}$).

Posterior estimates, simulation under indeterminacy			
	True values	Posterior estimates	
		Mean	90% probability interval
ψ_1	0.73	0.76	[0.71,0.81]
ψ_2	0.16	0.14	[0.02,0.26]
ρ_R	0.67	0.69	[0.65,0.75]
π^*	4.03	3.28	[2.02,4.56]
r^*	1.22	1.42	[1.13,1.72]
κ	0.86	0.77	[0.49,1.04]
τ^{-1}	1.61	1.89	[1.32,2.50]
ρ_g	0.77	0.76	[0.70,0.82]
ρ_z	0.78	0.77	[0.72,0.81]
σ_R	0.22	0.21	[0.20,0.22]
σ_g	0.24	0.23	[0.17,0.29]
σ_z	1.10	1.06	[0.98,1.14]
ρ_{gz}	0.46	0.49	[0.20,0.79]
σ_ν	0.24	0.25	[0.17,0.32]
$\rho_{R\nu}$	-0.19	-0.22	[-0.37,-0.07]
$\rho_{g\nu}$	0.15	0.22	[-0.25,0.72]
$\rho_{z\nu}$	-0.21	-0.19	[-0.38,0.02]

Table 4.4: Posterior distributions obtained by estimating the model using the simulation under indeterminacy.

4.2 Unknown region of determinacy

In this section, we assume that the region of determinacy, $|\psi^*| > 1$, is *unknown.* By considering this case, we show that our methodology can be used to study LRE models for which it is non-trivial to derive an analytic condition describing the region of determinacy. Thus, the approach allows a researcher to estimate medium- and large-scale LRE models that could potentially be characterized by indeterminacy. Our methodology allows the researcher to conduct Bayesian inference on the model parameters over the entire parameter space and to compute their posterior estimates which could potentially lie in both regions of determinacy and indeterminacy.

The assumption that the region of determinacy is unknown implies that it is no longer possible to impose $\alpha \equiv |\psi^*|$. To ensure that the Metropolis-Hastings algorithm explores the entire parameter space, we assume a uniform distribution over the interval $[0, 2]$ as a prior distribution for the parameter α.[25] Equivalently, we assume that there is an equal probability of making draws of α from the interval $[0,1)$ as well as from the interval $[1,2]$. Draws of α from $[1,2]$ combined with draws of the other parameters θ which satisfy the condition $|\psi^*| > 1$ ensure to solve the augmented representation under determinacy. Similarly, draws of α from $[0,1)$ combined with draws of the other parameters of interest θ such that $0 < |\psi^*| \leq 1$ ensure to solve the proposed representation under indeterminacy.[26]

Importantly, the same intuition described in Section 4.1 still holds. The Metropolis-Hastings algorithm makes draws of α and θ which could solve the augmented representation under determinacy and indeterminacy, and it compares the posterior obtained for draws in both regions. Having specified the prior for α, we estimate the augmented representation using the same two simulations of the data as in Section 4.1. We first estimate the augmented representation of the model using the data simulated under determinacy and the same prior distributions reported in Table 4.2.

The posterior distribution for the parameter α is plotted in Figure 4.2. Two remarks should be made. First, the posterior distribution is distributed over the interval $[1,2]$, thus providing evidence that the Metropolis-Hastings algorithm explores the entire parameter space and successfully recovers the information contained in the simulated data about model determinacy. Second, the posterior distribution approximates a uniform distribution over the same interval. This result is in line with the non-identifiability of the parameter α stated in Corollary 3. Finally, the posterior mean and 90% probability intervals of the parameters are the same as those reported in Table 4.3 when we assume that the region of determinacy is known. As in Section 4.1, the estimation procedure conducted on our

[25] The choice of the interval $[0,2]$ is arbitrary. For any value $0 < a \leq 1$, it is sufficient to specify an interval $[-a+1, a+1]$ as the domain of the uniform distribution.

[26] The virtue of using a continuous distribution for α and treating it as any other parameter of the model is that the algorithm can be easily implemented in Dynare. However, the efficiency of the algorithm could be improved by using a discrete distribution for α given that the only thing that matters is if this parameter is inside or outside the unit circle. Furthermore, the MCMC algorithm could be modified to allow for the possibility that whenever the augmented model does not have a solution, the value of α is flipped.

Posterior distribution of parameter α (determinacy)

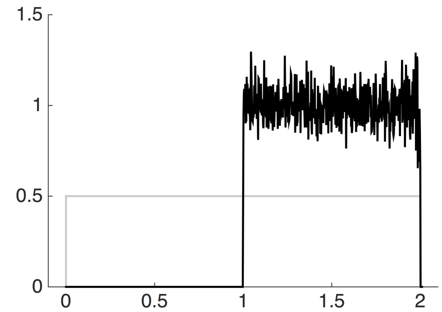

Figure 4.3: Posterior distribution of the parameter α under determinacy.
The grey line represents the prior distribution for the parameter α. The black line is the posterior distribution.

augmented representation recovers the true parameter values. Also in this case, the results are independent of the initial parametrization used to start the algorithm. Regardless of whether the starting parametrization is in the 'correct' region, the estimation algorithm successfully recovers the true parameter values used for the simulations. However, the speed of the convergence for the parameter estimates might be affected.

The estimation of the augmented representation using simulated data under indeterminacy delivers a mirrored posterior distribution for the parameter α (Figure 4.2). In this case, the posterior distribution of the parameter α is distributed over the interval $[0, 1)$ and it closely resembles a uniform distribution over the same interval due to its non-identifiability. As for the simulation under determinacy, we obtain the same posterior mean and the 90% probability interval as for the case of a known region of determinacy reported in Table 4.4. Hence, also when we assume that the region is unknown to the researcher, we recover the true parameter values by estimating the augmented representation.

4.3 Indeterminacy in the 1970s

This section provides an example on how to implement our methodology when using real data. We retain the assumption that the researcher does not know the region of determinacy, and we show that our method enables the algorithm to jump across the regions of determinacy and indeterminacy, thus facilitating the search for the *global* maximum in the marginal data

Posterior distribution of parameter α (indeterminacy)

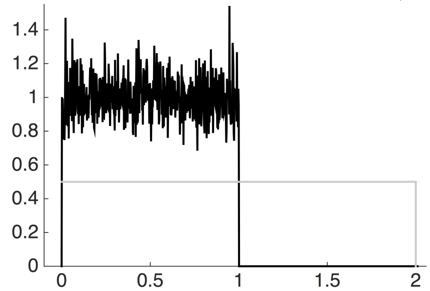

Figure 4.4: Posterior distribution of the parameter α under indeterminacy.
The grey line represents the prior distribution for the parameter α. The black line is the posterior distribution.

density.

We consider both the model and the data that Lubik and Schorfheide (2004) use to test for indeterminacy in U.S. monetary policy. The model is described by equations (4.1)$\sim$(4.6) at the beginning of Section 4 and, as previously explained, we append the process in (4.8) to obtain the augmented representation that we propose. Finally, equation (4.10) presents the measurement equations that link the endogenous variables of the model to the data. In the following, we focus on the data for the pre-Volcker period (1960Q1 - 1979Q2) since Lubik and Schorfheide (2004) show that during this period the monetary authority did not respond aggressively enough to changes in inflation, thus not suppressing self-fulfilling inflation expectations.

We proceed by starting the algorithm from initial conditions in both regions of the parameter space and by allowing for a large number of draws.[27] We verified that this approach guarantees the proper convergence of the posterior distributions for any initial parametrization by repeating this estimation exercise 100 times and successfully recovering the same posterior estimates in each case. Figure 4.3 reports the posterior distribution for both ψ_1 and α which clearly favor the indeterminate model regardless of the initial values for the parameters.[28]

[27]In particular, we run two chains of 1,000,000 each and discard the first half of the draws.

[28]The prior that we used for the parameter ψ_1 is the same as in Table 5 and is defined on both regions

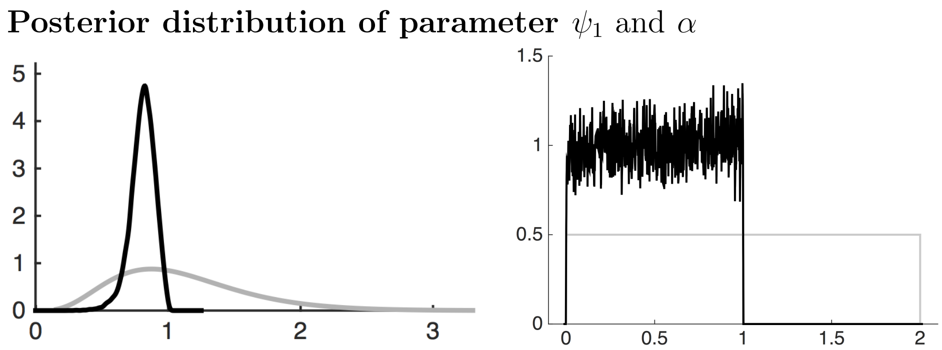

Figure 4.5: Posterior distribution of parameter ψ_1 and α. Initial parametrization $\psi_1 = 0.9$. The grey line represents the prior distribution for the parameter α. The black line is the posterior distribution.

Table 4.3 reports the corresponding posterior mean and 90% probability interval of the model parameters. As expected, the estimates obtained using our procedure are in line with the empirical results in Lubik and Schorfheide (2004) that we report in the first column.[29]

5 Tips for implementation

In this section, we present some suggestions for the practical implementation of our method.

Convergence. We repeat the estimation of the model of Lubik and Schorfheide (2004) by using parameters in the "wrong" region of the parameter space and considering only a few (200,000) draws to show the importance of checking convergence before interpreting the estimation results. Figure 5 reports the posterior distribution for the parameter ψ_1 and α obtained for an initial parametrization close to the Taylor Principle (i.e. we set $\psi_1 = 1.1$). At first glance, the posterior distribution of the parameter ψ_1 would appear to be bimodal. This is consistent with the fact that the proposed augmented representation allows the

of determinacy and indeterminacy. This is important for the possibility of the posterior estimates to lie in either region since having a prior distribution which assigns zero probability to either one would also imply that the posterior would have no mass in the same region.

[29]The minor difference in the point estimate of the posterior mean for the correlation between demand and supply shocks, ρ_{gz}, derives from the prior distribution that we assume for this parameter. While Lubik and Schorfheide (2004) assume a normal prior centered at 0 and with standard deviation 0.4, we assume a flat, uniform distribution over the interval $(-1, 1)$. Nevertheless, both in Lubik and Schorfheide (2004) and in this paper, the estimate of ρ_{gz} is not statistically significant.

Posterior estimates			
	LS estimates	Posterior estimates	
		Mean	90% probability interval
α	-	0.95	[0.90,0.99]
ψ_1	0.77	0.73	[0.56,0.90]
ψ_2	0.17	0.16	[0.01,0.30]
ρ_R	0.60	0.67	[0.48,0.86]
π^*	4.28	4.03	[1.87,6.06]
r^*	1.13	1.22	[0.64,1.78]
κ	0.77	0.86	[0.45,1.26]
τ^{-1}	1.45	1.61	[0.93,2.27]
ρ_g	0.68	0.77	[0.66,0.88]
ρ_z	0.82	0.78	[0.68,0.88]
σ_R	0.23	0.22	[0.19,0.25]
σ_g	0.27	0.25	[0.16,0.32]
σ_z	1.13	1.10	[0.93,1.27]
ρ_{gz}	0.14	0.47	[-0.04,0.95]
σ_ν	-	0.24	[0.16,0.33]
$\rho_{R\nu}$	-	-0.19	[-0.65,0.27]
$\rho_{g\nu}$	-	0.15	[-0.40,0.71]
$\rho_{z\nu}$	-	-0.21	[-0.55,0.14]

Table 4.5: Posterior distributions obtained by estimating the model using the data from Lubik and Schorfheide (2004).
The terms "-" indicate that the estimates are not directly comparable.

Metropolis-Hastings algorithm to visit both regions of the parameter space. At the same time, the posterior distribution for the parameter α is very similar to the prior distribution, which is specified as a uniform distribution over the interval $[0, 2]$. Such a result is just the other side of coin of the posterior for ψ_1 since the algorithm explores both regions by considering draws of α which are within as well as outside the unit circle.

A researcher should then verify the occurrence of either of the following two circumstances. This bimodal distribution could arise because the log-likelihood is highly discontinuous between the two regions. In this case, the algorithm could have jumped towards the region where the peak of the posterior lies, without having spent a significant time there. In other words, convergence has not occurred yet. Alternatively, if the log-likelihood function varies smoothly between the two regions of the parameter space, the posterior distribution

Posterior distribution of parameter ψ_1 and α

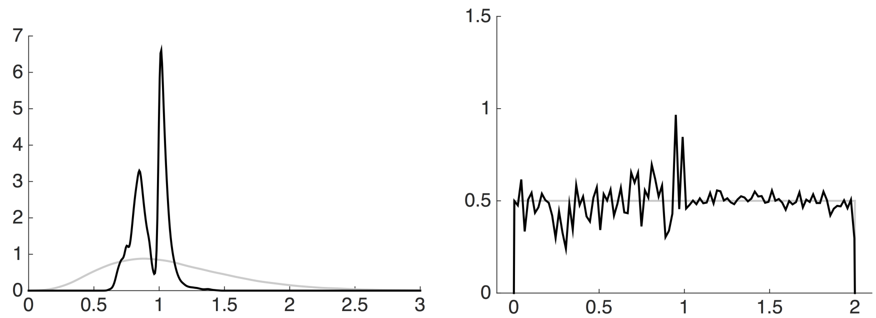

Figure 5.1: Posterior distribution of parameter ψ_1 and α. Initial parametrization $\psi_1 = 1.1$. The grey line represents the prior distribution for the parameter α. The black line is the posterior distribution.

plotted in Figure 5 could be the result of the algorithm traveling across the two regions multiple times.

We therefore recommend the researcher to analyze the draws of the parameter α which have been accepted during the MCMC algorithm. By inspecting the behavior of the auxiliary parameter α, a researcher can detect if the algorithm reached convergence or not. We report the draws that we obtained during our exercise in Figure 5.2. After approximately 40,000 draws of α in the region of determinacy (i.e. outside the unit circle), the algorithm jumps to the indeterminate region and never visits the determinacy region again.

Draws of the parameter α

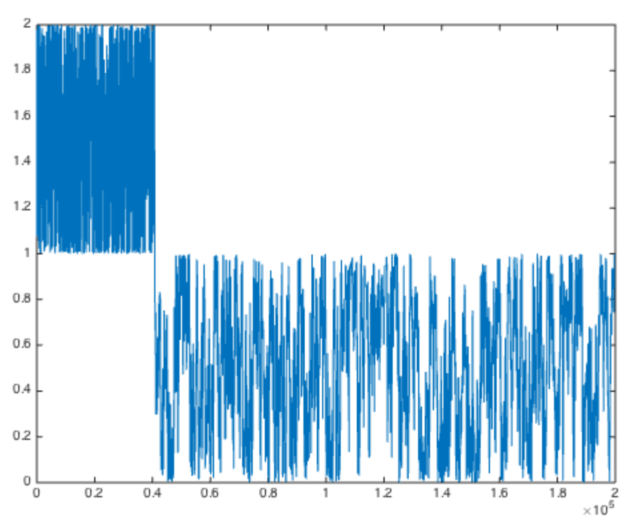

Figure 5.2: Sequence of draws for α given an initial parametrization $\psi_1 = 1.1$.

Figure 5 and 5.2 suggest that we are in the first case, for which the log-likelihood function is highly discontinuous at the boundary between the two regions. Therefore, the researcher should repeat the estimation exercise, increase the number of draws, and make sure that the parameter α stabilizes on one region of the parameter space. Under different circumstances, the researcher could face the second scenario, for which the log-likelihood function transitions smoothly between the two regions. In this case, the parameter α would repeatedly transition between the two areas of the parameter space and could be used to infer the probability attached to determinacy.

Only (in)determinacy. In some cases, a researcher might want to estimate the model exclusively under determinacy or exclusively under indeterminacy. Our approach easily accommodates this need. If the researcher is only interested in the solution under determinacy, the parameter vector of alpha should be chosen in a way to guarantee stationarity of the auxiliary process (for example, fixing all values of the alphas to 2). Furthermore, all parameters that are relevant only under indeterminacy could be fixed to zero or any other constant, given that they do not affect the fit of the model under determinacy. If instead the researcher is only interested in estimating the model under indeterminacy, the parameters of the auxiliary process can be chosen in a way to guarantee that the correct number of explosive roots are provided. In this case, the parameters describing the properties of the sunspot disturbances should also be estimated.

Model comparison. A researcher might also be interested in comparing the fit of the model under determinacy and under indeterminacy. Model comparison can be conducted by using standard techniques, such as the harmonic mean estimator proposed by Geweke (1999a). If the researcher is interested in comparing the same model under determinacy and under indeterminacy, we recommend the following procedure that adapts the approach used by Lubik and Schorfheide (2004):

1. Estimate the model under determinacy by fixing the parameter(s) alpha to a value larger than one in a way that the model is solved only under determinacy. Note that

in this case all parameters that pertain to the solution under indeterminacy, such as the volatility of the sunspot shocks, *should* be restricted to zero (or any other constant). This restriction avoids penalizing the model for extra parameters that do not affect its fit under determinacy.

2. Estimate the model under indeterminacy by fixing the parameter(s) alpha to a value smaller than one in a way that the model is solved only under indeterminacy. Note that in this case all parameters that pertain to the solution under indeterminacy, such as the volatility of the sunspot shocks, should be estimated.

3. Use standard methods to compare the fit of the model under determinacy with the fit of the same model under indeterminacy.

6 Conclusions

In this paper, we propose a generalized approach to solve and estimate LRE models over the entire parameter space. Our approach accommodates both cases of determinacy and indeterminacy and it does not require the researcher to know the analytic condition describing the region of determinacy or the degrees of indeterminacy.

When a LRE model is characterized by m degrees of indeterminacy, our approach augments it by appending m autoregressive processes whose innovations are linear combinations of a subset of endogenous shocks and a vector of newly defined sunspot shocks. The resulting augmented representation embeds both the solution which is obtained under determinacy using standard solution methods and that delivered by solving the model under indeterminacy using the approach of Lubik and Schorfheide (2003) and equivalently Farmer et al. (2015). We provide an analytical example for the theoretical result using a canonical NK model.

We finally apply our methodology to the NK model in Lubik and Schorfheide (2004). We simulate two series of data under the assumption of model determinacy and indeterminacy and we then estimate our augmented representation for both cases in which the region of determinacy is known or unknown to the researcher. In both case, the parameters used to generate the data are correctly recovered independently of the initial parametrization. This shows that our method is suitable for the estimation of medium- and large-size DSGE model for which the determinacy region is generally unknown. This feature of the solution method is used by Nicolò (2017) to study the possibility of multiple solutions in Smets and Wouters (2007).

Part III

Keynesian Economics Without the Phillips Curve

United States macroeconomic data are well described by co-integrated non-stationary time series (Nelson and Plosser, 1982). This is true, not just of data that are growing such as GDP, consumption and investment. It is also true of data that are predicted by economic theory to be stationary such as the unemployment rate, the output gap, the inflation rate and the money interest rate, (King et al., 1991; Beyer and Farmer, 2007b).[30]

The dominant New Keynesian paradigm is a three-equation model that explains persistent high unemployment by positing that wages and prices are 'sticky' (Galí, 2008; Woodford, 2003). Sticky-price models have difficulty generating enough persistence to understand the near unit root in unemployment data, as do models of the monetary transmission mechanism that assume sticky information (Mankiw and Reis, 2007) or rational inattention, (Sims, 2001a).[31]

Farmer (2012a) provides an alternative explanation of persistent high unemployment that we refer to as the Farmer Monetary (FM) model.[32] The FM model differs from the three-equation NK model by replacing the Phillips curve with the *belief function* (Farmer, 1993), a new fundamental that has the same methodological status as preferences and technology.

[30] A bounded random variable, such as the unemployment rate, cannot be a random walk over its entire domain. We view the I(1) assumption to be an approximation that is valid for finite periods of time.

[31] We prefer to avoid the assumption of menu costs (Mankiw, 1985) or price rigidity (Christiano et al., 2005; Smets and Wouters, 2007), because our reading of the evidence as surveyed by Klenow and Malin (2010), is that prices at the micro level are not sticky enough to explain the properties of monetary shocks in aggregate data. The approach we follow here generates *permanent* equilibrium movements in the unemployment rate that are consistent with a unit root, or near unit root, in U.S. unemployment data.

[32] Farmer and Konstantin Platonov (Farmer and Platonov, 2016) build on this idea to explain the relationship between the FM model and alternative interpretations of the textbook IS-LM model (Mankiw, 2010) on which modern New-Keynesian models are based.

In the FM model, search frictions lead to the existence of multiple steady state equilibria, and the steady-state unemployment rate is determined by aggregate demand.

The FM model displays both static and dynamic indeterminacy. Static indeterminacy means there are many possible equilibrium steady-state unemployment rates. Dynamic indeterminacy means there are many dynamic equilibrium paths, all of which converge to a given steady state. We resolve both forms of indeterminacy with a belief function that pins down a unique rational expectations equilibrium.

The structural properties of the FM model translate into a critical property of its reduced form. Appealing to the Engle-Granger Representation Theorem (Engle and Granger, 1987), we show that the FM model's reduced form is a co-integrated Vector Error Correction Model (VECM). The inflation rate, the output gap, and the federal funds rate, are non-stationary but display a common stochastic trend. The fact that our model is described by a VECM, rather than a VAR, implies that it displays hysteresis. In the absence of stochastic shocks, the model's steady-state depends on initial conditions.

The FM model was introduced by Farmer (2012a) in a paper in which he discussed the limitations of the NK model and proposed the FM model as an alternative. Our paper extends his work in three directions.[33]

First, we study the role of monetary feedback rules in stabilizing inflation, the output gap and the unemployment rate in the FM model.[34] It is well known that the NK model has a unique determinate steady state when the central bank reacts aggressively to stabilize inflation, a concept that Michael Woodford (2003) refers to as the *Taylor principle*. We develop the FM analog of the Taylor principle and we show that it does not hold in the U.S.

[33] Related papers to our current work are those of Farmer (2012b,a), Plotnikov (2012, 2013) and Farmer and Platonov (2016). Farmer (2012a) develops the basic three-equation model that we work with here and he discusses the philosophy that distinguishes his approach from the NK model. Farmer (2012b) developed the labor market theory that accounts for persistent unemployment and Plotnikov (2013, 2012) adds investment and capital accumulation. Farmer and Platonov (2016) extend the theoretical model of Farmer (2012b) by adding money.

[34] Because there is a one-to-one mapping between the output gap and the difference of unemployment from its natural rate, we will move freely in our discussion between these two concepts.

data either before or after 1980.

Second, we use the fact that our analog of the Taylor Principle fails to hold in the U.S. data to explain the observation that monetary shocks have real effects. Unlike the NK model, which assumes that prices are exogenously sticky, we explain the real effects of nominal shocks as an endogenous equilibrium response to nominal shocks which is enforced by the properties of the belief function.

Third, we exploit the property of static indeterminacy to explain why the unemployment rate has a (near) unit root in U.S. data. Our model resolves both dynamic indeterminacy and static indeterminacy by introducing beliefs about future nominal income growth as a new fundamental. We assume that individuals form expectations about future nominal income growth and we model these expectations as a martingale as in Farmer (2012a).

1 Relationship with Previous Literature

Our paper is connected with an empirical literature that studies the medium term persistence of business cycles. This includes the work of Robert King, Charles Plosser, James Stock and Mark Watson (1991), Diego Comin and Mark Gertler (2006), King and Watson (1994), and Andreas Beyer and Farmer (2007b).

Importantly, this literature finds that the unemployment rate is highly persistent and one cannot reject the hypothesis that the unemployment rate is a random walk. Viewed through the lens of neoclassical or New Keynesian theoretical models, the persistence of the unemployment rate is a supply side phenomenon. Something must be changing in either technology or preferences to cause permanent changes in the natural rate of unemployment.[35]

The supply-side approach is not the only way to interpret the fact that unemployment is

[35]This is the interpretation of Robert Gordon (Gordon, 2013), who argues that unemployment is non-stationary because the natural rate of unemployment is a random walk. Because the natural rate of unemployment is associated with the solution to a social planning optimum, if persistent unemployment were caused by a permanent increase in the natural rate of unemployment, high persistent unemployment would, at least to a first approximation, be socially optimal. That is possible, but in our view, implausible.

persistent. As Christopher Sims demonstrated in his seminal paper on Vector Autoregressions (Sims, 1980), rational expectations models are typically under-identified.[36] That fact leads to an important question: Is the persistent slow-moving component of the unemployment rate caused by aggregate supply shocks, or is it caused by aggregate demand shocks? This paper is part of a growing literature that provides theory and evidence in favor of the demand-side explanation for the persistence of high unemployment following a recession.[37]

2 The Structural Forms of the NK and FM Models

In Section 2 we write down the two structural models that form the basis for our empirical estimates in Section 7. These models have two equations in common. One of these is a generalization of the NK IS curve that arises from the Euler equation of a representative agent. The other is a policy rule that describes how the Fed sets the fed funds rate. The two common equations of our study are described below.[38]

2.1 Two Equations that the NK and FM Models Share in Common

We assume the log of potential real GDP grows at a constant rate and we estimate this series by regressing the log of real GDP on a constant and a time trend. The residual series is our empirical analog of the output gap. The FM model implies that the output gap is

[36]Building on that idea, Beyer and Farmer (2004) provided an algorithm to construct families of models, all of which have the same likelihood in a given data set. Some of the models generated by their method lead to a unique determinate equilibrium; others lead to an indeterminate equilibrium driven by self-fulfilling beliefs. Both classes of theoretical models have the same likelihood.

[37]Lawrence Summers (Summers, 2014) has recently resurrected a term *secular stagnation* coined by Alvin Hansen (Hansen, 1939) to refer to the fact that the economy may be stuck in a period of permanent under-employment equilibrium and Gauti Eggertsson and Neil Mehrotra (Eggertsson and Mehrotra, 2014) have formalized Hansen's mechanism in an overlapping generations framework. My own previous work provides a complete internally consistent explanation of secular stagnation that is consistent with the fact that the stock market and the unemployment rate are cointegrated random walks (Farmer, 2012b, 2013; Farmer and Platonov, 2016). Olivier Blanchard and Summers (Blanchard and Summers, 1986, 1987) provided an alternative explanation.

[38]Our discussion in sections 2 and 3 closely follows Farmer (2012a).

non-stationary and cointegrated with the CPI inflation rate and the federal funds rate. The NK model implies that the output gap is stationary.

In Equations (2.1) and (2.2), y_t is our constructed output gap measure, R_t is the federal funds rate and π_t is the CPI inflation rate. The term $z_{d,t}$ is a demand shock, $z_{R,t}$ is a policy shock and $z_{s,t}$ is a random variable that represents the Fed's estimate of potential GDP.[39]

$$ay_t - a\mathbb{E}_t(y_{t+1}) + [R_t - \mathbb{E}_t(\pi_{t+1})]$$
$$= \eta\left(ay_{t-1} - ay_t + [R_{t-1} - \pi_t]\right) + (1-\eta)\rho + z_{d,t}. \quad (2.1)$$

$$R_t = (1-\rho_R)\bar{r} + \rho_R R_{t-1} + (1-\rho_R)\left[\lambda\pi_t + \mu\left(y_t - z_{s,t}\right)\right] + z_{R,t}. \quad (2.2)$$

Equation (2.1) is a generalization of the dynamic IS curve that appears in standard representations of the NK model. In the special case when $\eta = 0$ this equation can be derived from the Euler equation of a representative agent.[40] An equation of this form for the general case when $\eta \neq 0$ can be derived from a heterogeneous agent model (Farmer, 2016) where the lagged real interest rate captures the dynamics of borrowing and lending between patient and impatient groups of people. In the case when $\eta = 0$, the parameter a is the inverse of the intertemporal elasticity of substitution and ρ is the time preference rate.

Equation (2.2) is a *Taylor Rule* (Taylor, 1999) that represents the response of the monetary authority to the lagged nominal interest rate, the inflation rate and the output gap. The monetary policy shock, $z_{R,t}$, denotes innovations to the nominal interest rate caused by unpredictable actions of the monetary authority. The parameters ρ_R, λ and μ are policy elasticities of the fed funds rate with respect to the lagged fed funds rate, the inflation rate and the output gap.

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[39]More precisely, $z_{s,t}$ is the Fed's estimate of the deviation of the log of potential GDP from a linear trend.

[40]See for example Galí (2008), or Woodford (2003).

2.2 Two Equations that Differentiate the Two Models

The third equation of the NK model is given by

$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \phi \left(y_t - z_{s,t} \right). \tag{3.a}$$

Here, β is the discount rate of the representative person and ϕ is a compound parameter that depends on the frequency of price adjustment.[41] Since β is expected to be close to one, we will impose the restriction $\beta = 1$ when discussing the theoretical properties of the model. This restriction implies that the long-run Phillips curve is vertical. If instead, $\beta < 1$, the NK model has an upward sloping long-run Phillips curve in inflation-output gap space. An extensive literature derives the NK Phillips curve from first principles, see for example Galí (2008), based on the assumption that frictions of one kind or another prevent firms from quickly changing prices in response to changes in demand or supply shocks.

In contrast to the NK Phillips curve, the FM model is closed by a belief function (Farmer, 1993). The functional form for the belief function that we use in this study is described by Equation (3.b),

$$\mathbb{E}_t \left[x_{t+1} \right] = x_t, \tag{3.b}$$

where $x_t \equiv \pi_t + (y_t - y_{t-1})$ is the growth rate of nominal GDP.

The belief function is a mapping from current and past observable variables to probability distributions over future economic variables. In the functional form we use here, it asserts that agents' forecast about future nominal GDP growth will equal current nominal GDP growth; that is, nominal GDP growth is a martingale. Farmer (2012a) has shown that this specification of beliefs is a special case of adaptive expectations in which the weight on

[41] In the NK model, the discount parameter β that appears in the Phillips curve is related to the parameter ρ that appears in the IS curve by the identity $\beta \equiv \frac{1}{1+\rho}$. We did not impose that restriction in our estimates and we thank the the editor of this issue for asking for clarification on this point. If we *had* imposed it, our results in favor of the FM model would have been even stronger since the restriction does not hold exactly in the estimates reported in Tables 3 and 4.

current observations of GDP growth is equal to 1.[42]

In the FM model, the monetary authority chooses whether changes in the current growth rate of nominal GDP will cause changes in the expected inflation rate or in the output gap. Importantly, these changes will be permanent. The belief function, interacting with the policy rule, selects how demand and supply shocks are distributed between permanent changes to the output gap, and permanent changes to the expected inflation rate.

3 The Steady-State Properties of the Two Models

In this section we compare the theoretical properties of the non-stochastic steady-state equilibria of the NK and FM models. The NK model has a unique steady-state equilibrium. The FM model, in contrast, has a continuum of non-stochastic steady-state equilibria. Which of these equilibria the economy converges to depends on the initial condition of a system of dynamic equations. This property is known as *hysteresis*.[43]

Rather than treat the multiplicity of steady state equilibria as a deficiency, as is often the case in economics, we follow Farmer (1993) by defining a new fundamental, the *belief function*. When the model is closed in this way, equilibrium uniqueness is restored and every sequence of shocks is associated with a unique sequence of values for the three endogenous variables.

We begin by shutting down shocks and describing the theoretical properties of the steady-state of the NK model. The values of the steady-state inflation rate, interest rate and output gap in the NK model are given by the following equations

$$\bar{\pi} = \frac{\phi(\bar{r} - \rho)}{\phi(1 - \lambda) - \mu(1 - \beta)}, \qquad \bar{R} = \rho + \bar{\pi}, \qquad \bar{y} = \bar{\pi}\frac{(1 - \beta)}{\phi}. \tag{3.1}$$

[42] Farmer (2012a) allowed for a more general specification of adaptive expectations and he found that the data favor the special case we use here.

[43] This analysis reproduces the discussion from Farmer (2012a) and we include it here for completeness.

When $\beta < 1$, the long-run Phillips curve, in output gap-inflation space, is upward sloping. As β approaches 1, the slope of the long-run Phillips curve becomes vertical and these equations simplify as follows,

$$\bar{\pi} = \frac{(\bar{r} - \rho)}{(1 - \lambda)}, \qquad \bar{R} = \rho + \bar{\pi}, \qquad \bar{y} = 0. \tag{3.2}$$

For this important special case, the steady state of the NK model is defined by Equations (3.2).

Contrast this with the steady state of FM model, which has only two steady state equations to solve for three steady state variables. These are given by the steady state version of the IS curve, Equation (2.1), and the steady state version of the Taylor Rule, Equation (2.2).

The FM model is closed, not by a Phillips Curve, but by the belief function. In the specific implementation of the belief function in this paper we assume that beliefs about future nominal income growth follow a martingale. This equation does not provide any additional information about the non-stochastic steady state of the model because the same variable, steady-state nominal income growth, appears on both sides of the equation.

Solving the steady-state versions of equations (2.1) and (2.2) for $\bar{\pi}$ and $\bar{R}$ as a function of $\bar{y}$ delivers two equations to determine the three variables, $\bar{\pi}$, $\bar{R}$ and $\bar{y}$.

$$\bar{\pi} = \frac{(\bar{r} - \rho)}{(1 - \lambda)} + \frac{\mu}{(1 - \lambda)}\bar{y}, \qquad \bar{R} = \rho + \bar{\pi}. \tag{3.3}$$

The fact that there are only two equations to determine three variables implies that the steady-state of the FM model is under determined. We refer to this property as *static indeterminacy*. Static indeterminacy is a source of endogenous persistence that enables the FM model to match the high persistence of the unemployment rate in data.

In standard economic models, the approximate system that describes how the variables evolve through time is a linear difference equation with a point attractor. In the absence of stochastic shocks, the model economy converges asymptotically to this point. In the FM model the approximate system that describes how the variables evolve through time is a linear difference equation with a one dimensional line as its attractor. In the absence of stochastic shocks, the model economy converges asymptotically to a point on this line, but which point it converges to depends on the initial condition. The reduced form representation of the FM model is a VECM, as opposed to a VAR.

An implication of the static indeterminacy of the model is that policies that affect aggregate demand have permanent long-run effects on the output gap and the unemployment rate. In contrast, the NK model incorporates the Natural Rate Hypothesis, a feature which implies that demand management policy cannot affect real economic activity in the long-run.

4 The FM Analog of the Taylor Principle: A Simple Case

In this section we discuss the NK Taylor Principle and we derive an analog of this principle for the FM model. For both the NK and FM models we study the special case of $\rho_R = 0$, and $\eta = 0$. The first of these restrictions sets the response of the Fed to the lagged interest rate to zero. The second restricts the IS curve to the representative agent case. These restrictions allow us to generate, and compare, analytical expressions for the Taylor Principle in both models.

The special cases of Equations (2.1) and (2.2) are given by

$$ay_t = aE_t(y_{t+1}) - (R_t - \mathbb{E}_t(\pi_{t+1})) + \rho + z_{d,t}, \tag{1'}$$

and

$$R_t = \bar{r} + \lambda \pi_t + \mu (y_t - z_{s,t}) + z_{R,t}. \tag{2'}$$

The Taylor Principle directs the central bank to increase the federal funds rate by more than one-for-one in response to an increase in the inflation rate. When the Taylor Principle is satisfied, the dynamic equilibrium of the NK model is locally unique. When that property holds, we say that the unique steady state is locally determinate (Clarida et al., 1999).

When the central bank responds only to the inflation rate, the Taylor principle is sufficient to guarantee local determinacy. When the central bank responds to the output gap as well as to the inflation rate, a sufficient condition for the NK model to be locally determinate is that

$$\left|\lambda + \frac{1-\beta}{\phi}\mu\right| > 1. \tag{4.1}$$

For the important special case of $\beta = 1$ this reduces to the familiar form of the Taylor principle (Woodford, 2003).

In Appendix A we derive this result analytically and we compare it with the dynamic properties of the FM model. There we establish that for the special case of logarithmic preferences, that is when $a = 1$, a sufficient condition for local determinacy in the FM model is,

$$\left|\frac{\lambda}{\lambda - \mu}\right| > 1. \tag{4.2}$$

This is our FM analog of the Taylor principle. In the usual case when λ and μ are positive, it requires the interest-rate response of the central bank to changes in inflation to be greater than the output-gap response. When this condition holds, each element of the *set* of steady state equilibria of the model is dynamically determinate.

5 The FM Analog of the Taylor Principle: The General Case

When the representative agent has CRA preferences with $a \neq 1$, the FM version of the Taylor principle is more complicated and we are unable to find an analytic expression except in the case when $\lambda = \mu$.[44] In this special case, the Taylor Principle fails whenever

$$a < 1 + \frac{\lambda}{2}. \tag{5.1}$$

Although the parameter restriction, $\lambda = \mu$, is unlikely to hold in practice, it does give us an indication of whether or not the FM Taylor principle is likely to hold outside of the case of logarithmic preferences. The answer to that question is no. Consider, as an example, the special case when $\lambda = \mu = 0.7$. For this parametrization, the determinacy condition fails when a is larger than 1.35. Since estimates of a in data are typically larger than 2, it seems likely that failure of the Taylor Principle will be the normal case. Indeed, that conjecture is verified by our empirical estimates. When we allow λ and μ to differ and we estimate them using Bayesian techniques, our estimated model displays dynamic indeterminacy for positive values of a that are greater than, but much closer to, one.

In Figure 5.1 we set three key parameters to their estimated values of $\eta = 0.89$, $\rho = 0.021$, and $\rho_R = 0.98$ and we plot the roots of the system as functions of the risk-aversion parameter a. This matrix always has a unit root and a root of zero.[45] The determinacy condition requires that the remaining two roots must both be greater than one in absolute value. The figure shows that for our estimated parameter values, one root falls below unity in absolute value for values of a greater than 1.004.

We conclude from our analysis of the roots that for plausible parametrizations, the FM model displays dynamic as well as static indeterminacy and this conclusion is confirmed by our empirical estimates, described in Section 7, in which we freely estimate a to be equal to 3.8.

The conjunction of static and dynamic indeterminacy provides two sources of endogenous persistence. Static indeterminacy implies that the output gap contains an I(1) component.

[44] We provide a derivation of the analytic result in Appendix B.

[45] Since the unit root and the root of zero do not depend on the parameter values, we do not display them in Figure 5.1.

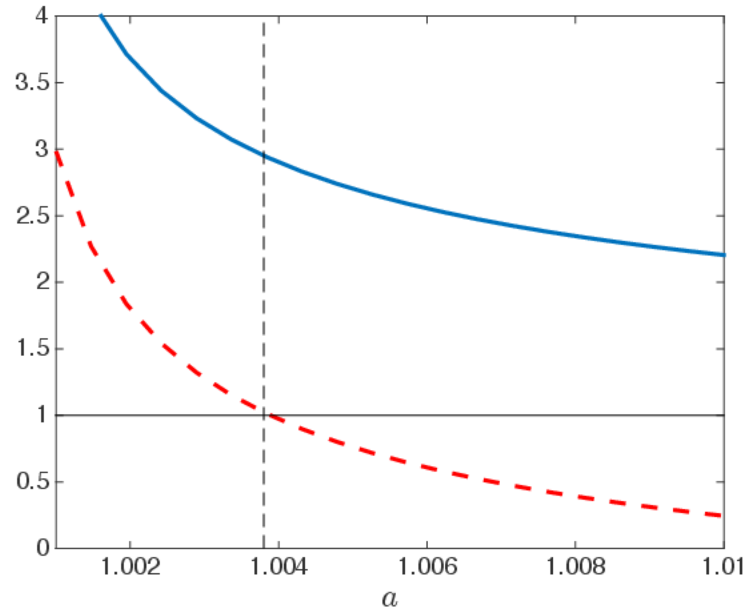

Figure 5.1: Characteristic roots as a function of a: $\lambda = 0.76$, $\mu = 0.75$

Instead of converging to a point in interest-rate-inflation-output gap space, the data converge to a one-dimensional linear manifold. Dynamic indeterminacy implies that the fed funds rate, the inflation rate and the unemployment rate display persistent deviations from this manifold.

The fact that the model displays dynamic indeterminacy allows us to explain why prices appear to move slowly in data. In response to an increase in the Fed Funds Rate it is the output gap, not the inflation rate, that bears the burden of adjustment.

In a model with fully flexible prices and a locally unique equilibrium, the current and expected future price respond on impact to maintain a constant real interest rate. In the FM model, there is no artificial barrier to price adjustment, but people believe, rationally, that it is quantities and not prices that will respond to an interest rate surprise. Prices are posted one period in advance and an unexpected increase in the Fed Funds Rate leads to a self-fulfilling increase in the real interest rate that dies out asymptotically as future prices respond to the interest rate shock.

In the FM model, prices are not sticky in the sense that there is a cost or barrier to price adjustment. They are sticky because people believe, correctly, that future prices will validate their decision to demand fewer goods and services in response to an increase in the money interest rate.

6 Solving the NK and FM Models

6.1 Finding the Reduced forms of the Two Models

Sims (2001b) showed how to write a structural DSGE model in the form

$$\Gamma_0 X_t = C + \Gamma_1 X_{t-1} + \Psi \varepsilon_t + \Pi \nu_t \tag{6.1}$$

where $X_t \in \mathbb{R}^n$ is a vector of variables that may or may not be observable. Using the following definitions, the NK and FM models can both be expressed in this way,[46]

$$X_t \equiv \begin{bmatrix} y_t \\ \pi_t \\ R_t \\ \mathbb{E}_t(y_{t+1}) \\ \mathbb{E}_t(\pi_{t+1}) \\ z_{d,t} \\ z_{s,t} \end{bmatrix}, \quad \varepsilon_t \equiv \begin{bmatrix} z_{R,t} \\ \varepsilon_{d,t} \\ \varepsilon_{s,t} \end{bmatrix}, \qquad \nu_t = \begin{bmatrix} \nu_{1,t} \\ \nu_{2,t} \end{bmatrix} \equiv \begin{bmatrix} y_t - \mathbb{E}_{t-1}(y_t) \\ \pi_t - \mathbb{E}_{t-1}(\pi_t) \end{bmatrix}. \tag{6.2}$$

The shocks ε_t are called *fundamental* and the shocks ν_t are *non-fundamental*. These shocks are equal to the one-step ahead forecast errors of y_t and π_t and, in models with a unique determinate steady-state, they are determined endogenously as functions of the fundamental shocks, ε_t. By exploiting a property of the generalized Schur decomposition (Gantmacher, 2000) Sims provided an algorithm, GENSYS, that determines if there exists a VAR of the

[46]We assume, in our estimation, that $z_{d,t}$ and $z_{s,t}$ may be auto-correlated but we restrict $z_{R,t}$ to be i.i.d. For this reason, the innovations to $z_{d,t}$ and $z_{s,t}$ appear in ε_t along with the realized value of $z_{R,t}$.

form

$$X_t = \hat{C} + G_0 X_{t-1} + G_1 \varepsilon_t, \tag{6.3}$$

such that all stochastic sequences $\{X_t\}_{t=1}^{\infty}$ generated by this equation also satisfy the structural model, Equation (6.1).[47] To guarantee that solutions remain bounded, all of the eigenvalues of G_0 must lie inside the unit circle. When a solution of this kind exists, we refer to it as a reduced form of (6.1).

GENSYS reports on whether a reduced form exists and, if it exists, whether it is unique. The algorithm eliminates unstable generalized eigenvalues of the matrices $\{\Gamma_0, \Gamma_1\}$ by finding expressions for the non-fundamental shocks, ν_t, as functions of the fundamental shocks, ε_t. When there are too few unstable generalized eigenvalues, there are many candidate reduced forms.

For the case of multiple candidate reduced forms, Farmer et al. (2015) show how to redefine a subset of the non-fundamental shocks as new fundamental shocks. For example, if the model has one degree of indeterminacy, one may define a vector of *expanded fundamental shocks*, $\hat{\varepsilon}_t$,

$$\hat{\varepsilon}_t \equiv \begin{bmatrix} \varepsilon_t \\ \nu_{2,t} \end{bmatrix}. \tag{6.4}$$

The parameters of the variance-covariance matrix of expanded fundamental shocks are fundamentals of the model that may be calibrated or estimated in the same way as the parameters of the utility function or the production function.

We assume that prices are subject to an independent sunspot shock that is uncorrelated with the innovations to the other three fundamental shocks. This assumption forces shocks to the policy rule to be transmitted contemporaneously to the output gap and it enables the FM model to explain a monetary transmission mechanism in which nominal shocks are transmitted to prices slowly over time.

[47] The generalized Schur decomposition exploits the properties of the generalized eigenvalues of the matrices $\{\Gamma_0, \Gamma_1\}$.

To solve and estimate both the NK and FM models, we use an implementation of GENSYS, (Sims, 2001b) programmed in DYNARE (Adjemian et al., 2011), to find the reduced form associated with any given point in the parameter space. We use the Kalman filter to generate the likelihood function and a Markov Chain Monte Carlo algorithm to explore the posterior.

6.2 An Important Implication of the Engle-Granger Representation Theorem

The reduced form of both the NK and FM models is a Vector Autoregression with the form of Equation (6.3). We reproduce that equation below.

$$X_t = \hat{C} + G_0 X_{t-1} + G_1 \varepsilon_t. \tag{6.3$'$}$$

Robert Engle and Clive Granger (1987) showed how to rewrite a Vector-Autoregression in the equivalent form

$$\Delta X_t = \hat{C} + \hat{\Pi} X_{t-1} + G_1 \varepsilon_t, \tag{6.5}$$

where $X_t \in \mathbb{R}^n$. If the matrix $\hat{\Pi}$ has rank n, this system of equations has a well defined non-stochastic steady state, $\bar{X}$, defined by shutting down the shocks and setting $X_t = \bar{X}$ for all t. $\bar{X}$ is defined by the expression,

$$\bar{X} = -\hat{\Pi}^{-1} \hat{C}. \tag{6.6}$$

When $\hat{\Pi}$ has rank $r < n$, it can be written as the product of an $n \times r$ matrix α and an $r \times n$ matrix $\beta^\top$,

$$\hat{\Pi} = \alpha \beta^\top. \tag{6.7}$$

The rows of α are referred to as loading factors, and the columns of β are called co-integrating vectors.[48]

When $\hat{\Pi}$ has reduced rank there is no steady state and in the absence of stochastic shocks the sequence X_t will converge to a point on an $n-r$ dimensional linear subspace of $\mathbb{R}^n$ that depends on the initial condition X_0.

The NK model has a unique steady state and its Engle-Granger representation leads to a matrix $\hat{\Pi}$ with full rank. In contrast, the FM model has multiple steady states and its Engle-Granger representation leads to a matrix $\hat{\Pi}$ with reduced rank. It follows that the reduced form of the FM model is a VECM as opposed to a VAR.

7 Estimating the Parameters of the NK and FM Models

In this section we estimate the NK and FM models. Both models share equations (2.1) and (2.2) in common. We reproduce these equations below for completeness.

$$
\begin{aligned}
&ay_t - a\mathbb{E}_t(y_{t+1}) + [R_t - \mathbb{E}_t(\pi_{t+1})] \\
&\qquad = \eta\left(ay_{t-1} - ay_t + [R_{t-1} - \pi_t]\right) + (1-\eta)\rho + z_{d,t}. \quad (1)
\end{aligned}
$$

$$R_t = (1-\rho_R)\bar{r} + \rho_R R_{t-1} + (1-\rho_R)\left[\lambda\pi_t + \mu\left(y_t - z_{s,t}\right)\right] + z_{R,t}. \quad (2)$$

For the NK model these equations are supplemented by the Phillips curve, Equation (3.a),

$$\pi_t = \beta\mathbb{E}_t[\pi_{t+1}] + \phi\left(y_t - z_{s,t}\right), \quad (3.a)$$

and for the FM model they are supplemented by the belief function, Equation (3.b),

$$\mathbb{E}_t\left[x_{t+1}\right] = x_t. \quad (3.b)$$

[48] The co-integrating vectors are not uniquely defined; they are linear combinations of the steady state equations of the non-stochastic model.

We assume in both models that the demand and supply shocks follow autoregressive processes that we model with equations (7.1) and (7.2),

$$z_{d,t} = \rho_d z_{d,t-1} + \varepsilon_{d,t}, \tag{7.1}$$

$$z_{s,t} = \rho_s z_{s,t-1} + \varepsilon_{s,t}. \tag{7.2}$$

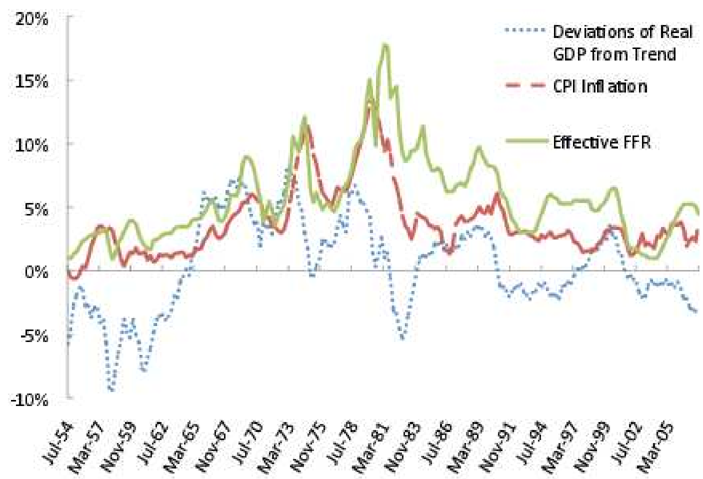

Figure 7.1: U.S. data

Source: FRED, Federal Reserve Bank of St. Louis.

Figure 7.1 plots the data that we use to compare the models. We use three time series for the U.S. over the period from 1954Q3 to 2007Q4: the effective Federal Funds Rate, the CPI inflation rate and the percentage deviation of real GDP from a linear trend.

To estimate the models, we used a Markov-Chain Monte-Carlo algorithm, implemented in DYNARE (Adjemian et al., 2011). Formal tests reject the null of parameter constancy over the entire period. Beyer and Farmer (2007b) find evidence of a break in 1980 and we know from the Federal Reserve Bank's own website (of San Francisco, 2003) that the Fed pursued a monetary targeting strategy from 1979Q3 through 1982Q3. For this reason, and in line with previous studies (Clarida et al., 2000a; Lubik and Schorfheide, 2004; Primiceri, 2005), we estimated both models over two separate sub-periods.

Our first sub-period runs from 1954Q3 through 1979Q2. The beginning date is one year after the end of the Korean war; the ending date coincides with the appointment of Paul Volcker as Chairman of the Federal Reserve Board. We excluded the period from 1979Q3 through 1982Q4 because, over that period, the Fed was explicitly targeting the growth rate of the money supply. In 1983Q1, it reverted to an interest rate rule.

Our second sub-period runs from 1983Q1 to 2007Q4. We ended the sample with the Great Recession to avoid potential issues arising from the fact that the federal funds rate hit a lower bound in the beginning of 2009 and our linear approximation is unlikely to fare well for that period.

Table 1.A: Prior distribution, common model parameters

Name	Range	Density	Mean	Std. Dev.	90% interval
a	R^+	*Gamma*	3.5	0.50	[2.67,4.32]
ρ	R^+	*Gamma*	0.02	0.005	[0.012,0.028]
η	$[0,1)$	*Beta*	0.85	0.10	[0.65,0.97]
$\bar{r}$	R^+	*Uniform*	0.05	0.029	[0.005,0.095]
ρ_R	$[0,1)$	*Beta*	0.85	0.10	[0.65,0.97]
μ	R^+	*Gamma*	0.70	0.20	[0.41,1.06]
ρ_d	$[0,1)$	*Beta*	0.80	0.05	[0.71,0.87]
ρ_s	$[0,1)$	*Beta*	0.90	0.05	[0.81,0.97]
σ_R	R^+	*Inverse Gamma*	0.01	0.003	[0.005,0.015]
σ_d	R^+	*Inverse Gamma*	0.01	0.003	[0.005,0.015]
σ_{η_2}	R^+	*Inverse Gamma*	0.005	0.003	[0.002,0.010]
ρ_{ds}	[-1,1]	*Uniform*	0	0.58	[-0.9,0.9]
ρ_{dR}	[-1,1]	*Uniform*	0	0.58	[-0.9,0.9]
ρ_{sR}	[-1,1]	*Uniform*	0	0.58	[-0.9,0.9]
β	$[0,1)$	*Beta*	0.97	0.01	[0.95,0.98]
ϕ	R^+	*Gamma*	0.50	0.20	[0.22,0.87]

Table 1.B: Prior distribution for each sample period

Name	Range	Density	Mean	Std. Dev.	90% interval
Pre-Volcker					
λ	R^+	*Gamma*	0.9	0.50	[0.26,1.85]
σ_s	R^+	*Inverse Gamma*	0.1	0.03	[0.06,0.15]
Post-Volcker					
λ	R^+	*Gamma*	1.1	0.50	[0.42,2.02]
σ_s	R^+	*Inverse Gamma*	0.01	0.005	[0.005,0.019]

Table 1.A summarizes the prior parameter distributions that we used in this procedure for those parameters that were the same in both sub-samples. The table reports the prior shape, mean, standard deviation and 90% probability interval. Table 1.B presents the prior distributions for parameters that were different in the two subsamples. These were λ, the policy coefficient for the interest rate response to the inflation rate, and σ_s, the standard deviation of the supply shock.

We set $\lambda = 0.9$ in the first sub-period and $\lambda = 1.1$ in the second. We chose these values because Lubik and Schorfheide (2004) found that policy was indeterminate in the first period and determinate in the second. These choices ensure that our priors are consistent with these differences in regimes.

We set the standard deviation of σ_s to 0.1 in the pre-Volcker sample and 0.01 in the post-Volcker sample. We made this choice because earlier studies (Primiceri, 2005; Sims and Zha, 2006) found that the variance of shocks was higher in the post-Volcker sample, consistent with the fact that there were two major oil-price shocks in this period.

We restricted the parameters of the policy rule to lie in the indeterminacy region for the pre-Volcker period and the determinacy region for the post-Volcker. Those restrictions are consistent with Lubik and Schorfheide (2004) who estimated a NK model, pre- and post-Volcker, and found that the NK model was best described by an indeterminate equilibrium in the first sub-period. Our priors for a, λ and μ place the FM model in the indeterminacy region of the parameter space for *both* sub-samples.

To identify the NK model in the pre-Volcker period, and for the FM model in both sub-periods, we chose a pre-determined price equilibrium. We selected that equilibrium by choosing the forecast error

$$\nu_{2,t} \equiv \pi_t - \mathbb{E}_{t-1}[\pi_t]$$

as a new fundamental shock and we identified the variance covariance matrix of shocks by setting the covariance of $\nu_{2,t}$ with the other fundamental shocks, to zero.[49]

The results of our estimates are reported in Tables 2, 3 and 4. Table 2 reports the logarithm of the marginal data densities and the corresponding posterior model probabilities under the assumption that each model has equal prior probability. These were computed using the modified harmonic mean estimator proposed by Geweke (1999b). In Tables 3 and 4 we present parameter estimates for the pre-Volcker period (1954Q3-1979Q2) and the post-Volcker period, (1983Q1-2007).

Table 2: Model comparison

		FM model	NK model
Pre-Volcker (54Q3-79Q2)	Log data density	1023.24	1017.26
	Posterior Model Prob (%)	100	0
Post-Volcker (83Q1-07Q4)	Log data density	1136.22	1121.42
	Posterior Model Prob (%)	100	0

We find that, in both subsamples, the posterior model probability is 100% in favor of the FM model. In words, the data strongly favor the VECM representation over the VAR.

[49]We denote the corresponding standard deviation by σ_{ν_2}.

Table 3: **Posterior estimates, Pre-Volcker** (54Q3-79Q2)				
	FM model		NK model	
	Mean	90% probability interval	Mean	90% probability interval
a	3.80	[3.11,4.46]	3.70	[2.91,4.49]
ρ	0.020	[0.012,0.027]	0.017	[0.010,0.023]
η	0.87	[0.83,0.92]	0.76	[0.63,0.89]
$\bar{r}$	0.051	[0.014,0.093]	0.043	[0.002,0.079]
ρ_R	0.94	[0.91,0.97]	0.98	[0.97,0.99]
λ	0.80	[0.22,1.34]	0.45	[0.17,0.73]
μ	0.74	[0.44,1.03]	0.56	[0.28,0.84]
ρ_d	0.76	[0.69,0.83]	0.80	[0.72,0.88]
ρ_s	0.95	[0.92,0.98]	0.78	[0.71,0.86]
σ_R	0.007	[0.006,0.008]	0.008	[0.007,0.009]
σ_d	0.011	[0.009,0.013]	0.011	[0.007,0.014]
σ_s	0.097	[0.059,0.133]	0.059	[0.043,0.073]
σ_{η_2}	0.003	[0.003,0.004]	0.003	[0.002,0.004]
ρ_{Rd}	0.79	[0.64,0.95]	-0.06	[-0.30,0.17]
ρ_{Rs}	-0.53	[-0.80,-0.26]	0.59	[0.43,0.76]
ρ_{ds}	-0.79	[-0.94,-0.65]	0.11	[-0.22,0.47]
β	n/a	n/a	0.98	[0.97,0.99]
ϕ	n/a	n/a	0.07	[0.04,0.09]

The dynamic properties of the FM model depend on the value of the parameter a. We tried restricting this parameter to be less than 1, a restriction that places the FM model in the determinacy region of the parameter space. We found that the posterior for a model that imposes this restriction was clearly dominated by allowing a to lie in the indeterminacy region.

Table 4: Posterior estimates, Post-Volcker (83Q1-07Q4)				
	FM model		NK model	
	Mean	90% probability interval	Mean	90% probability interval
a	4.23	[3.46,4.99]	3.62	[2.87,4.35]
ρ	0.020	[0.012,0.028]	0.023	[0.016,0.029]
η	0.93	[0.88,0.99]	0.93	[0.89,0.98]
$\bar{r}$	0.045	[0.024,0.064]	0.008	[0.001,0.016]
ρ_R	0.75	[0.63,0.88]	0.93	[0.89,0.97]
λ	0.50	[0.17,0.80]	1.39	[1.04,1.70]
μ	0.85	[0.52,1.18]	0.64	[0.34,0.92]
ρ_d	0.78	[0.71,0.85]	0.63	[0.55,0.71]
ρ_s	0.90	[0.84,0.97]	0.94	[0.91,0.98]
σ_R	0.004	[0.004,0.005]	0.006	[0.005,0.006]
σ_d	0.008	[0.006,0.009]	0.007	[0.005,0.009]
σ_s	0.022	[0.008,0.038]	0.011	[0.008,0.014]
σ_{η_2}	0.005	[0.004,0.006]	n/a	n/a
ρ_{Rd}	-0.47	[-0.67,-0.27]	0.27	[0.10,0.45]
ρ_{Rs}	0.88	[0.77,0.99]	0.20	[0.01,0.40]
ρ_{ds}	-0.62	[-0.89,-0.34]	0.70	[0.56,0.85]
β	n/a	n/a	0.97	[0.95,0.99]
ϕ	n/a	n/a	0.26	[0.11,0.41]

Until recently, standard software packages such as DYNARE (Adjemian et al., 2011) or GENSYS (Sims, 2001b) would either crash or return an error flag in response to a parameter vector for which the model is indeterminate. In our empirical work, we avoid this problem by drawing on the work of Farmer et al. (2015) who show how to deal with indeterminate models by redefining a subset of the non-fundamental errors as fundamentals.

Table 3 reports the estimated parameters of both the FM and NK models. For both of these models, the parameter estimates place the model in the indeterminacy region and, in both cases, we resolved the indeterminacy by selecting an equilibrium in which the covariance parameters of shocks to the inflation surprise with the other fundamentals shocks was set to zero.

Table 4 reports the posterior estimates for the post-Volcker period (1983Q1-2007Q4). For this sample period, the FM estimates place the model in the region of dynamic indeterminacy and, once again, we resolved the indeterminacy by selecting a pre-determined price equilibrium. In contrast, the posterior means of the NK model satisfy the Taylor Principle, thus guaranteeing that the equilibrium of NK model is locally unique.

We find differences in the policy parameters $\bar{r}$, and μ and large significant differences in λ, and ρ_R. In line with previous studies (Primiceri, 2005; Sims and Zha, 2006), we find that the estimated volatility of the shocks dropped significantly after the Volcker disinflation.

In Section 8 we provide further insights on the role that these changes played in affecting the relationship between inflation rate, output gap and nominal interest rate.

8 What Changed in 1980?

There is a large literature that asks: Why do the data look different after the Volcker disinflation? At least two answers have been given to that question. One answer, favored by Sims and Zha (2006), is that the primary reason for a change in the behavior of the data before and after the Volcker disinflation is that the variance of the driving shocks was larger in the pre-Volcker period. Primiceri (2005) finds some evidence that policy also changed but his structural VAR is unable to disentangle changes in the policy rule from changes in the private sector equations.

Previous work by Canova and Gambetti (2009) explains the reduction in volatility after 1980 as a consequence of better monetary policy. But when Lubik and Schorfheide (2004) estimate a NK model over two separate sub-periods they find significant difference across regimes, not only in the policy parameters, but also in their estimates of the private sector parameters. That leads to the following question. Can the FM model explain the change in the behavior of the data before and after 1980 in terms of a change only in the policy parameters? To answer that question, we estimated five alternative models. The results are reported in Table 5.

In Model 1, Fully unrestricted, we estimated all the parameters of the FM model separately for the two sub-periods. In Model 2, Policy and shocks, we allowed the variances of the shocks and the parameters of the policy rule to change across sub-periods, but we constrained the parameters of the IS curve to be the same. In Models 3, Shocks only, we allowed only the variances of the shocks to change and in Model 4, we allowed only the Policy Rule parameters to change. Finally, in Model 5, we restricted all of the parameters to be the same in both sub-periods.

The results in Table 5 indicate that the specification in which policy parameters and shocks are allowed to differ explains the data almost as well as the fully unrestricted model specification. But as soon as we restrict either the policy parameters or the shocks to be the same, the explanatory power of the FM model drops substantially. With the exception of Model 2, Policy and shocks, all of the restrictions are clearly rejected.

Table 5: Model specifications

	Log data density	Posterior model prob
Fully unrestricted	2159.48	-
Policy and shocks	2159.39	47.7%
Shocks only	2141.56	0%
Policy only	2121.42	0%
Fully restricted	2113.25	0%

Why was the post-Volcker regime relatively benign? In line with previous studies, we find that both good policy and good luck had a part to play. The post-Volcker period, leading up to the Great Recession, was associated with fewer large shocks and with no large negative supply shocks of the same order of magnitude as the oil price shocks of 1973 and 1978. It was also associated with a change in the policy rule followed by the Fed. What we add to previous studies is a model in which our estimates of the structural private sector parameters remain invariant across both regimes. The Fed changed its behavior; households did not.

9 Conclusions

The FM model gives a very different explanation of the relationship between inflation, the output gap and the federal funds rate from the conventional NK approach. It is a model where demand and supply shocks may have permanent effects on employment and inflation and our empirical findings demonstrate that this model fits the data better than the NK alternative. The improved empirical performance of the FM model stems from its ability to account for persistent movements in the data.

In the FM model, beliefs about nominal income growth are fundamentals of the economy. Beliefs select the equilibrium that prevails in the long-run and monetary policy chooses to allocate shocks to permanent changes in inflation expectations or permanent deviations of output from its trend growth path.

Our findings have implications for the theory and practice of monetary policy. Central bankers use the concept of a time-varying natural rate of unemployment before deciding when and if to raise the nominal interest rate. The difficulty of estimating the natural rate arises, in practice, because the economy displays no tendency to return to its natural rate. That fact has led to much recent skepticism about the usefulness of the Phillips curve in policy analysis. Although we are sympathetic to the Keynesian idea that aggregate demand determines employment, we have shown in this paper that it is possible to construct a 'Keynesian economics' without the Phillips curve.

Part IV

Monetary Policy, Expectations and Business Cycles in the U.S. Post-War Period

This paper examines the interactions between monetary policy and the formation of expectations to explain U.S. business cycle fluctuations in the post-war period. Previous studies mainly use medium-scale New-Keynesian (NK) models and assume that the central bank implemented an 'active' monetary policy that systematically stabilizes inflation and output growth during the entire post-war period.

This assumption does not reconcile with the data. From the late 1950s through the 1970s, the U.S. economy experienced high volatility, and inflation was high and rising. Assuming an 'active' monetary policy does not allow to account for propagation mechanisms based on the de-anchoring of inflationary expectations in response to structural shocks. However, it simplifies the construction of the solution in such models.

I estimate the conventional medium-scale NK model by Smets and Wouters (2007) (henceforth SW), in which I relax the key assumption that the central bank pursued an 'active' monetary policy both before and after 1979. If monetary policy is passive, the model is indeterminate and characterized by multiple equilibrium paths. Two features of the model become relevant to explain the persistence and volatility of the data. First, the propagation of structural shocks depends on self-fulfilling expectations that generate an additional source of persistence. Second, unexpected changes in expectations constitute non-fundamental 'sunspot' disturbances that generate an additional source of uncertainty.

I find four main results. First, the conduct of U.S. monetary policy changed in the post-war period. Monetary policy was passive between 1955 and 1979, while it pursued an active inflation targeting since 1984. Compared to previous studies that use medium-scale models, this result rejects the imposed assumption that monetary policy was active before 1979.

Second, the evidence of a passive monetary policy from 1955 to 1979 substantially affects the explanation of U.S. business cycles over this period. According to the estimated model, fundamental productivity and cost shocks were the primary drivers of the run-up in the inflation rate from the early 1960s to 1979. Positive technology shocks in the 1960s de-anchored inflation expectations from the central bank's long-run target and generated persistent *inflationary* pressures via self-fulfilling expectations.[50] Mark-up shocks account for the sudden inflationary episodes related to the oil crisis during the 1970s, while they are not significant drivers of the rise in inflation during the 1960s.

Third, the high volatility of inflation and output growth before 1979 was caused by fundamental disturbances and not by sunspot shocks. In a passive monetary policy regime, non-fundamental shocks potentially lead to additional macroeconomic instability. By contrast, the estimation of the SW model shows that non-fundamental sunspot shocks were not significant drivers of volatility between 1955 and 1979.

Finally, I revisit the question on the sources of the reduction in U.S. macroeconomic volatility from the 1980s to 2007. I investigate whether the observed decrease in volatility is explained by a more active monetary policy since the early 1980s, as opposed to smaller structural shocks. Based on the SW model, I find that the reduction in macroeconomic uncertainty was a combination of both a change in monetary policy to a more active stance *and* a lower volatility of the shocks.

To solve the medium-scale model of SW with a passive monetary policy, I use the methodology developed in Bianchi and Nicolò (2017), which simplifies technical complexities that hamper the implementation of existing solution methods to medium-scale models (Lubik

[50]This result is supported by the empirical evidence documented by Fernald (2014a) and Gordon (2000) among others, who argue that the U.S. economy experienced a period of exceptional growth in productivity since World War II until the early 1970s.

and Schorfheide, 2003, Farmer et al., 2015).

To the best of my knowledge, this paper is the first study that quantitatively investigates the role of self-fulfilling expectations and non-fundamental disturbances for U.S. macroeconomic instability prior to 1979 in the context of a medium-scale model. Previous studies that allow for indeterminacy of U.S. monetary policy mainly adopt small-scale NK models and rationalize the empirical properties of the data before 1979 with a passive monetary policy (Clarida et al., 2000b, Lubik and Schorfheide, 2004).

The adoption of a medium-scale model provides two advantages. First, a richer *dynamic* and *stochastic* structure could explain the macroeconomic volatility and inflation persistence before 1979, even when monetary policy is *active*. This explanation could overturn the results in previous studies that adopted small-scale models (Beyer and Farmer, 2007a). Second, the richer structure constitutes a suitable framework to study the quantitative implications for business cycle fluctuations.

The rest of the paper is organized as follows. Section 1 highlights the contributions of the paper to the related literature. Section 2 motivates the adoption of medium-scale models to properly assess the role of U.S. monetary policy to explain business cycles. Section 3 describes the main features of the SW model and the data used to conduct the estimation of the model using Bayesian techniques. Section 4 explains the methodology developed in Bianchi and Nicolò (2017) and its implementation to construct and estimate the SW model allowing for indeterminacy. Section 5 presents the findings. Section 6 concludes.

1 Related Literature

The paper contributes to five strands of the literature. First, it provides an interpretation of U.S. business cycle fluctuations in the United States based on the role of self-fulfilling expectations. Previous studies mainly abstract from the possibility of observing policies that lead to indeterminate outcomes (Bianchi, 2013, Fernandez-Villaverde et al., 2010, Del Negro and Eusepi, 2011, Bianchi and Ilut, 2017). The contribution of this paper is to quantify the implications of a passive monetary policy for U.S. business cycle fluctuations in the post-war period. Under such regime, the propagation of structural shocks is more persistent due to the formation of self-fulfilling expectations. This mechanism identifies different determinants of business cycles. The upward trend in the inflation rate observed since the early 1960s is due to persistent technology shocks that generated strong economic activity and self-fulfilling inflationary expectations. Moreover, I show that sunspot shocks play no quantitative role in explaining the volatility observed before 1979.

Second, a vast literature rationalizes the role of monetary policy for the behavior of the data in the post-war period using univariate or small-scale Linear Rational Expectations (LRE) models (Clarida et al., 2000b, Lubik and Schorfheide, 2004, Coibon and Gorodnichenko, 2011, Boivin and Giannoni, 2006, Yasuo Hirose and Zandweghe, 2017, Bhattarai et al., 2016). Their findings align and support the evidence that the monetary authority failed to implement an active inflationary targeting before 1979.[51] However, a richer dynamic and stochastic structure could suffice to explain the macroeconomic volatility and inflation persistence before 1979, even when monetary policy is *active* (Beyer and Farmer, 2007a).[52] This paper addresses this concern using a canonical medium-scale NK model and shows that earlier findings carry over to the SW model.

Third, the adoption of a medium-scale LRE model raises two technical complexities. First, the partition of the parameter space into a determinate and indeterminate region is unknown for richer models. Second, the construction of the indeterminate solution requires a substantial amount of coding using the existing solution methods (Lubik and Schorfheide,

[51] Alternative explanations for the run-up of U.S. inflation since the early 1960s relate to the possibility that policymakers overestimated potential output (Orphanides, 2002) and the persistence of inflation in the Phillips curve (Primiceri, 2006). In this paper, I focus on understanding the mechanisms through which the de-anchoring of inflation expectations due to structural shocks could have played a relevant role to explain the macroeconomic instability in the period prior to 1979.

[52] A closely related literature also discusses the concerns due to model misspecification for the empirical performance of Dynamic Stochastic General Equilibrium models and provides policy analysis approaches to deal with it (Del Negro et al., 2007, Del Negro and Schorfheide, 2009).

2003, Farmer et al., 2015). Given the technical complexities, a researcher commonly estimates a medium-scale model by restricting *a priori* the parameter space to the unique, determinate region (Smets and Wouters, 2007, Arias et al., 2017). In this paper, I implement the methodology we developed in Bianchi and Nicolò (2017) to relax this assumption and estimate the medium-scale model by SW over the *entire* parameter space. I find that the assumption imposed in SW is rejected for the period before 1979. Importantly, I show that the assumption has quantitative implications for the identification of the main drivers of U.S. business cycles.

Fourth, the paper contributes to the literature that studies the sources of the reduction in U.S. macroeconomic volatility from the 1980s to 2007. I investigate the validity of two prominent theories that have been advocated to explain this empirical phenomenon. First, several studies show that the behavior of the data changed due to a decrease in the variance of the shocks driving the economy in the period subsequent the Volcker disinflation (Sims and Zha, 2006, Primiceri, 2005, Justiniano and Primiceri, 2008, Alejandro Justiniano and Tambalotti, 2010, Alejandro Justiniano and Tambalotti, 2011). This strand of the literature considers that the reduction in volatility is not related to monetary policy and it can therefore be considered as "good luck". Second, the work of Clarida et al. (2000b) and Lubik and Schorfheide (2004) among others indicates that monetary policy acted more systematically since the 1980s, therefore suggesting a view related to the "good policy". In this paper, I find that the data supports both theories. Both a change in the conduct of monetary policy to a more active stance *and* a significant drop in the volatility of structural shocks account for the decrease in U.S. macroeconomic uncertainty.

Finally, the paper contributes to the literature that studies the empirical implications of dynamic indeterminacy.[53] The contributions of Farmer and Guo (1994) and Farmer and Guo (1995) focus on relevance of sunspot shocks to explain business cycle fluctuations. More

[53]A second generation of models in the literature about indeterminacy relates to the possibility of observing multiple steady states for a given model. In this paper, I will refer to indeterminacy only as the dynamic properties of the model in the neighborhood of the unique steady state of a model.

recently, Lubik and Schorfheide (2004) empirically evaluate the possibility that monetary policy could lead to indeterminate outcomes and Bhattarai et al. (2016) enrich this analysis by accounting for a non-trivial interaction between monetary and fiscal policy. This paper considers the richer dynamic and stochastic structure of the SW model to empirically study the implications of dynamic indeterminacy for U.S. business cycles in the post-war period.

2 Reasons for the Adoption of Medium-scale Models

Several studies focus on the conduct of U.S.monetary policy in the post-war period by adopting univariate and small-scale models. Clarida et al. (2000b) estimate a monetary policy reaction function and therefore address the question using a univariate structural model. Lubik and Schorfheide (2004) (henceforth LS) test for indeterminacy in U.S. monetary policy during the post-war period by considering a conventional three-equation NK model.

However, two advantages arise with the adoption of richer models. Section 2.1 discusses an identification problem that could potentially undermine and overturn the results obtained with parsimonious models. Section 2.2 provides insights on how the conduct of a passive monetary policy affects the propagation of fundamental shocks via the formation of self-fulfilling expectations and allows for non-fundamental sunspot shocks to affect the economy. In this paper, the adoption of the medium-scale model in SW allows to verify whether the results in earlier studies are susceptible to the modeling choice and to assess the quantitative implications of a passive monetary policy for U.S. business cycles.

2.1 Identification Problem

Previous studies that allow for indeterminacy in U.S. monetary policy mainly adopt small-scale NK models and rationalize the empirical properties of the data before 1979 with a passive monetary policy (Clarida et al., 2000b, Lubik and Schorfheide, 2004). If monetary policy is ‘passive’, two features of the model become relevant to explain the persistence in inflation dynamics and the high volatility of U.S. macroeconomic data over this period. First, the propagation of structural shocks depends on self-fulfilling expectations that generate an additional source of persistence. Second, unexpected changes in expectations constitute non-fundamental 'sunspot' disturbances that generate an additional source of uncertainty.

However, findings in earlier studies are potentially susceptible to the choice of parsimonious models (Beyer and Farmer, 2007a). Small-scale models impose restrictions on the structure of the underlying economy. By excluding richer models, the restrictions favor the result of a passive monetary policy since missing propagation mechanisms are misinterpreted as evidence of this conclusion. The identification problem relates to the possibility that a model with a richer *dynamic* and *stochastic* structure could explain the macroeconomic volatility and inflation persistence before 1979, even when monetary policy is *active*. Adopting the medium-scale NK model of SW allows to verify whether previous findings carry over to a richer structure.[54]

In the spirit of Beyer and Farmer (2007a), the following analytic example provides an intuition of the identification problem that an econometrician faces when testing for indeterminacy. Suppose that a researcher studies the dynamics of the inflation rate using two alternative univariate LRE models. One model explains current inflation only as a function of expected inflation as described by equation (2.1)

$$\pi_t = aE_t(\pi_{t+1}). \tag{2.1}$$

Since the endogenous variable is expectational, the model is well-specified when the

[54]LS acknowledge that their results are sensitive to model misspecification since missing propagation mechanisms would favor the result of model indeterminacy. Their robustness check consists in comparing the fit of a small-scale NK model for the Pre-Volcker period with a richer model to account for missing propagation mechanisms. However, the comparison is between two structurally different models and the robustness check could therefore be sensitive to the choice of which propagation mechanism are included in the richer model. In this paper, I am instead considering the SW model for both the determinate and the indeterminate regions, while aiming at reducing the identification problem that is inherent to the question by considering a medium-scale model.

associated one-step ahead forecast error is also defined

$$\eta_t \equiv \pi_t - E_{t-1}(\pi_t). \tag{2.2}$$

Considering the case of $|a| > 1$, the model is *indeterminate*, and any process for inflation and its expectation that takes the following form solves the univariate model in (2.1) and (2.2)

$$\begin{cases} \pi_t &= \lambda \pi_{t-1} + \eta_t, \\ \\ E_t\left(\pi_{t+1}\right) &= \lambda^2 \pi_{t-1} + \lambda \eta_t, \end{cases} \tag{2.3}$$

where $\lambda \equiv a^{-1} < 1$. According to this model, the dynamics are explained by lagged inflation rate, and the only source of volatility is the non-fundamental shock, η_t.

The alternative univariate LRE model considered by the econometrician describes current inflation as a function not only of expected inflation but also of lagged inflation and a fundamental shock, ε_t,

$$\pi_t = aE_t(\pi_{t+1}) + b\pi_{t-1} + \varepsilon_t. \tag{2.4}$$

Given the definition of the forecast error $\eta_t \equiv \pi_t - E_{t-1}(\pi_t)$, the dynamics of the model depend on the two roots of the model denoted by θ and λ.[55] When only one root is unstable, the model has a unique, *determinate* solution. By assuming without loss of generality that $|\theta| > 1$ and $|\lambda| < 1$, the solution of the determinate model is

$$\begin{cases} \pi_t &= \lambda \pi_{t-1} + \frac{(\lambda+\theta)}{\theta} \varepsilon_t, \\ \\ E_t\left(\pi_{t+1}\right) &= \lambda^2 \pi_{t-1} + \lambda \frac{(\lambda+\theta)}{\theta} \varepsilon_t. \end{cases} \tag{2.5}$$

The identification problem arises due to the observational equivalence of the two alter-

[55] It can be shown that the roots of the model are related to the structural parameters of the model as follows: $a = 1/(\lambda + \theta)$ and $b = \lambda\theta/(\lambda + \theta)$.

native models. Without further information about the true variance of the shocks η_t and ε_t, the *indeterminate* model in (2.3) and the *determinate* model in (2.5) are characterized by the same likelihood function.

However, the choice of a parsimonious structure affects the inference of the econometrician by erroneously favoring the indeterminate model in (2.3). Suppose that the true data generating process for the inflation rate is the richer, determinate model in (2.4). Also, suppose the researcher chooses a parsimonious dynamic structure such as in (2.1) where lagged inflation is omitted. The inference would therefore mistakenly lead the econometrician to conclude that the data is consistent with the dynamics of the indeterminate model in (2.3) due to the observational equivalence.

The identification problem suggests that the findings in earlier studies of a passive monetary policy before 1979 could be undermined by the choice of a parsimonious small-scale model. Abstracting from relevant propagation mechanisms and structural shocks would favor this result. The adoption of richer models allows to verify if earlier findings rely on the modeling choice.

In this paper, I consider the medium-scale model of SW to verify whether the findings of Clarida et al. (2000b) and LS carry over to a model with a richer dynamic structure. As presented in Section 5.1.1, I find that the data still supports the evidence of a passive monetary policy before 1979. I then argue in Section 5.2 and 5.3 that the distinctive propagation of structural shocks under such monetary policy regime is the feature of the indeterminate representation that the data favors. Finally, in Section 5.1.3, I also shed light on the debate of whether the Fed did not follow an active inflation targeting during the period after the 2001 slump and therefore generated economic conditions that led to the Great Recession. Importantly, I show that while the analysis conducted using a small-scale model suggests the latter interpretation of the events, using the SW model the data indicates that the monetary authority implemented an active policy.

2.2 Digging into the Mechanisms

The second advantage of adopting a medium-scale model such as SW is to provide a suitable framework to quantitatively assess the implications that a passive monetary policy has on the macroeconomy. In this section, I use a simple classical monetary model to show that if monetary policy is passive, the dynamic and stochastic properties of the model differ in two dimensions. First, the propagation of fundamental shocks through the economy differs due to the formation of self-fulfilling expectations in response to the shocks. Second, the model is subject to an non-fundamental sunspot disturbances. While small-scale models are not sufficiently detailed, medium-scale models account for richer transmission mechanisms and provide a quantitative assessment of the relative importance in the data.

To provide the intuition, I consider a classical monetary model described by the Fisher equation

$$R_t = r_t + E_t(\pi_{t+1}), \tag{2.6}$$

and the simple Taylor rule

$$R_t = \phi_\pi \pi_t, \tag{2.7}$$

where R_t and π_t denote the deviations of the nominal interest rate and the inflation rate from their target level. I assume that the real interest rate r_t is given and follows a mean-zero Gaussian i.i.d. distribution.[56] To properly specify the model, I also define the one-step ahead forecast error associated with the expectational variable, π_t, as

$$\eta_t \equiv \pi_t - E_{t-1}(\pi_t). \tag{2.8}$$

[56] In the classical monetary model, the real interest rate results from the equilibrium in labor and goods market and it depends on the technology shocks. I am considering an exogenous process for the technology shocks and therefore I take the process for the real interest rate as given.

Combining (2.6) and (2.7), I obtain the univariate model

$$E_t(\pi_{t+1}) = \phi_\pi \pi_t - r_t. \tag{2.9}$$

In this simple model, the monetary authority is active if it responds to changes in the inflation rate by more than one for one. By recalling the Taylor rule in (2.7), this condition can be equivalently expressed as $|\phi_\pi| > 1$. The solution in this region of the parameter space is said to be determinate, and it is obtained by solving forward equation (2.9) as follows,

$$\begin{aligned} \pi_t &= \frac{1}{\phi_\pi} E_t(\pi_{t+1}) + \frac{1}{\phi_\pi} r_t \\ &= \frac{1}{\phi_\pi} r_t, \end{aligned} \tag{2.10}$$

where the second equality is derived by recalling the assumptions on r_t. The strong response of the monetary authority ensures that inflation is pinned down as a function of the exogenous real interest r_t.

Consequently, $E_t(\pi_{t+1}) = 0$, so that the expectations that agents hold about the future inflation rate are constant at its steady-state. The determinate solution is therefore described by the following system,[57]

$$\begin{cases} \pi_t = \frac{1}{\phi_\pi} r_t, \\ \\ E_t(\pi_{t+1}) = 0. \end{cases} \tag{2.11}$$

Conversely, a passive monetary policy, $|\phi_\pi| \leq 1$, significantly affects the dynamic and stochastic properties of the model. The solution is obtained by combining the definition of the forecast error, η_t, with the univariate model in (2.9) as

[57]Also, note that this implies that $\eta_t = \pi_t - E_{t-1}(\pi_t) = \pi_t = \frac{1}{\phi_\pi} r_t$. Therefore, the non-fundamental shock η_t is endogenously determined as a function of the structural shock, r_t.

$$\pi_t = E_{t-1}(\pi_t) + \eta_t$$

$$= \phi_\pi \pi_{t-1} + \eta_t - r_{t-1}.$$

Expectations about future inflation are therefore described as,

$$E_t(\pi_{t+1}) = \phi_\pi \pi_t - r_t$$

$$= \phi_\pi^2 \pi_{t-1} + \phi_\pi \eta_t - (r_t + \phi_\pi r_{t-1}).$$

Therefore, the solution corresponds to the following system of equations

$$\begin{cases} \pi_t = \phi_\pi \pi_{t-1} + \eta_t - r_{t-1}, \\ E_t(\pi_{t+1}) = \phi_\pi^2 \pi_{t-1} + \phi_\pi \eta_t - (r_t + \phi_\pi r_{t-1}). \end{cases} \quad (2.12)$$

The comparison of the representations in (2.11) and (2.12) shows that a change in monetary policy substantially affects the properties of the model and the interpretation of business cycle fluctuations in at least two dimensions. First, the impact and transmission of the *same* structural shock, r_t, on the dynamics of the model differs between the two specifications. While under determinacy the inflation rate also follows an i.i.d. process, under indeterminacy the shock de-anchors agents' expectations from the central bank's long-run target and transmits via the formation of self-fulfilling inflation expectations.[58] This is clearly not the

[58]The inflation rate is not affected by the structural shock to the real interest rate whenever it is assumed that the real interest rate and the non-fundamental shock, η_t, are assumed to be uncorrelated. In a more general setting, the data could prefer a specification in which the correlation between structural shocks and non-fundamental shocks differs from zero.

case for the determinate solution where expectations are constant at the long-run inflation rate and play no role for the dynamics of the model.

Second, if monetary authority is passive, the economy is subject to an additional, non-fundamental disturbance related to unexpected changes in agents' expectations, η_t. The sunspot shock therefore provides an additional source of uncertainty which could potentially help the model in matching the high volatility of the data in the period prior to the appointment of Paul Volcker as the chairman of the Federal Reserve System. By solving and estimating the SW model using the methodology in Bianchi and Nicolò (2017), I assess the quantitative relevance of each of these two properties of the model, especially for the period before 1979 that previous work showed to be associated with a passive monetary policy. In Section 5.2 and 5.3, I argue that the feature that the data favors is the distinctive propagation mechanism that relies on the formation of self-fulfilling expectations, while sunspot shocks were not significant sources of uncertainty.

3 The Model and Data

Dynamic stochastic general equilibrium (DSGE) models are useful tools to conduct quantitative policy analysis. To this purpose, a branch of the literature focused on developing richer models that could provide a better match with the data. Based on the conventional three-equation NK model, the work by Smets and Wouters (2003) and Christiano et al. (2005) expands the framework to account for relevant frictions and shocks. The model presented in Smets and Wouters (2007) now constitutes the heart of the structural DSGE models that are adopted by most central banks in advanced economies. While the reader is referred to the original paper for the details about the derivation of the model, this section describes its relevant features as well as the measurement equations and the data used to estimate the model using Bayesian techniques.

The model contains both real and nominal frictions. On the real side, households are assumed to form habit in consumption. By renting capital services to firms, households also face an adjustment cost and optimally choose the capital utilization rate with an increasing cost. Firms incur a fixed cost in production and are subject to nominal price rigidities *à la Calvo*, while indexing the optimized price to past inflation. Similarly, the model displays nominal wage frictions that also allow for indexation to past wage inflation.

The economy follows a deterministic, balanced growth path along which seven shocks drive the dynamics of the model. Three shocks affect the demand-side of the economy. A risk premium shock affects the household's intertemporal Euler equation by impacting the spread between the risk-free rate and the return on the risky asset. The investment-specific shock has an effect on the investment Euler equation that the household considers when choosing the amount of capital to accumulate. The third demand-side shock is an exogenous spending shock that impacts the aggregate resource constraint. Similarly, the supply-side of the economy is subject to three shocks: a productivity shocks well as price and wage mark-up shocks. Finally, the monetary authority follows a Taylor rule as described in equation (3.1),

$$R_t = \rho R_{t-1} + (1-\rho)\left\{r_\pi \pi_t + r_y\left(y_t - y_t^p\right) + r_{\Delta y}\left[\left(y_t - y_t^p\right) - \left(y_{t-1} - y_{t-1}^p\right)\right]\right\} + \varepsilon_t^R. \quad (3.1)$$

The monetary authority chooses the nominal interest rate, R_t, by allowing for some degree of interest rate inertia as measured by the parameter ρ. Changes in the inflation rate, π_t, and the output gap, defined as the deviations of actual output from its fully flexible price and wage counterpart, also generate a response by the monetary authority. The Taylor rule also accounts for changes in the output gap, while any unexpected deviation in the policy instrument is defined as a monetary policy shock, ε_t^R.[59]

To estimate the model, I use Bayesian techniques and the measurement equations that relate the macroeconomic data to the endogenous variables of the model are defined in

[59] The model also assumes that the monetary policy shock follows an autoregressive process defined by $\varepsilon_t^R = \rho_R \varepsilon_{t-1}^R + u_t^R$, where $u_t^R \overset{iid}{\sim} \mathcal{N}\left(0, \sigma_R^2\right)$. The same assumption also holds for the other structural shocks of the model.

equation (3.2),

$$\begin{bmatrix} dlGDP_t \\ dlCONS_t \\ dlINV_t \\ dlWAG_t \\ lHours_t \\ dlP_t \\ FEDFUNDS_t \end{bmatrix} = \begin{bmatrix} \bar{\gamma} \\ \bar{\gamma} \\ \bar{\gamma} \\ \bar{\gamma} \\ \bar{l} \\ \bar{\pi} \\ \bar{R} \end{bmatrix} + \begin{bmatrix} y_t - y_{t-1} \\ c_t - c_{t-1} \\ i_t - i_{t-1} \\ w_t - w_{t-1} \\ l_t \\ \pi_t \\ R_t \end{bmatrix}, \tag{3.2}$$

where dl denotes the percentage change measured as log difference and l denotes the log. The observables are the seven macroeconomic quarterly U.S. macroeconomic time series used in SW, and they match the number of shocks that affect the economy. The series considered are: the growth rate in real GDP, consumption, investment and wages, log hours worked, inflation rate measured by the GDP deflator, and the federal funds rate.

The deterministic balanced growth path is defined in terms of four parameters: $\bar{\gamma}$, the quarterly trend growth rate common to real GDP, consumption, investment and wages; $\bar{l}$, the steady-state hours worked (normalized to zero); $\bar{\pi}$, the quarterly steady-state inflation rate; $\bar{R}$, the steady-state nominal interest rate. Hence, the measurement equations in (3.2) relate the macroeconomic time series with the corresponding endogenous variables of the model $\{y_t, c_t, i_t, w_t, l_t, \pi_t, R_t\}$, while accounting for a balanced growth path.

While the full sample of SW ends in the fourth quarter of 2004, I updated the time series and in Section 5.1.1 I estimate the model over three sub samples. The first period starts in 1955:4, which corresponds to one year after the end of the Korean War, and it ends in 1969:4, the date in which the chairmanship of William Martin terminates.[60] The second sample considers the chairmanships of both Arthur Burns and William Miller, and it spans from 1970:1 until 1979:2. As I argue in Section 5.1.1, it is relevant to distinguish the first

[60] As argued in the work of Bernanke and Blinder (1992) and Bernanke and Mihov (1998), the federal funds rate has been the main policy tool in the United States in the post-war period, even if the Federal Reserve varied its operational procedures.

two sub samples since, in line with the evidence documented by Fernald (2014a) among others, the second period is characterized by slower productivity growth, thus resulting into a distinct balanced growth path. Finally, the beginning of the third period corresponds to 1984:1, in which Kim and Nelson (1999) initially identify a structural break in the U.S. business cycle. The end is marked by the Great Recession in 2007:3.

4 Methodology

The adoption of medium-scale DSGE models to study the conduct of monetary policy raises technical complexities. First, to compare determinate and indeterminate model solutions, a researcher must be able to partition the parameter space into a determinate and indeterminate region. While this partition can be easily derived analytically for small-scale models, it is generally unknown for larger models. Second, the model could be characterized by regions of the parameters space associated with multiple degrees of indeterminacy, and the researcher has to test for the potential degrees of indeterminacy of the model.[61] Third, standard software packages do not allow for indeterminacy.[62]

The application of existing solution methods to deal with indeterminacy in medium-scale models requires a substantial amount of coding work and technical skills (Lubik and Schorfheide, 2003, Lubik and Schorfheide, 2004). In practice, most of the papers simply rule out the possibility of indeterminacy by estimating the model exclusively in the determinate region of the parameter space. Among others, SW also adopt this approach and assume *a priori* a unique, determinate solution of the model.

The work of Bianchi and Nicolò (2017) develops a new method to solve and estimate LRE models allowing for indeterminacy of the model solution. While the paper builds on Lubik

[61] A grid point method could be used to numerically identify the region of the parameter space associated with the indeterminate solution and the degrees of indeterminacy. However, this method does not provide a mapping between the dynamic properties of the model and its structural parameters.

[62] Examples of standard solution algorithm are the code developed by Sims (2001b), Gensys, the toolkit by Uhlig (1999) and the algorithm of Anderson and Moore (1985) among others.

and Schorfheide (2003, 2004) and Farmer et al. (2015), the novelty is to provide an approach that, using the information in the data, endogenously partitions the parameter space into the determinate and indeterminate region, and deals with the possibility of multiple degrees of indeterminacy. Hence, this methodology substantially simplifies the approach to test for indeterminacy in U.S. monetary policy. I show that the assumption is rejected before 1979 and monetary policy was passive, even when accounting for a richer model.

The method proposes to augment the original model with a set of auxiliary equations that are used to provide the adequate number of explosive roots in presence of indeterminacy. The augmented representation also introduces a non-fundamental sunspot shock to construct the solution under indeterminacy. The characterization of the full set of equilibria under indeterminacy is parametrized by the additional parameters related to the standard deviation of the sunspot shock and its covariance with the structural shock of the model.

This augmented representation provides three main advantages. First, it accommodates both the case of determinacy and indeterminacy, while considering the same augmented system of equations. In particular, the solution in this expanded state space, if it exists, is always determinate, and is identical to the indeterminate solution of the original model. The model can therefore be solved by using standard solution algorithms. Second, given that the method accommodates both the case of determinacy and indeterminacy, the researcher does not need to take a stance on which area of the parameter space she is interested in exploring. Finally, even when the region of determinacy is unknown as in the case of medium-scale models, the methodology allows the researcher to estimate the model without imposing *a priori* assumptions about the uniqueness of the equilibrium. Information contained in the data indicates whether an estimated model is characterized by a unique solution or by multiplicity of equilibrium paths.

While Section 4.1 provides a simple analytical example to explain the methodology developed in Bianchi and Nicolò (2017), Section 4.2 describes how I implement it to test for indeterminacy in U.S. monetary policy in the richer medium-scale model by SW.

4.1 Building the Intuition

I consider a simple analytical example to present the technical complexities that a researcher faces when dealing with indeterminacy and to provide an intuition for how the methodology developed in Bianchi and Nicolò (2017) simplifies the construction of the solution under indeterminacy. Recalling the classical monetary model in Section 2.2, I report below the corresponding univariate representation

$$E_t(\pi_{t+1}) = \phi_\pi \pi_t - r_t. \tag{4.1}$$

As previously described, the solution to this model depends on the conduct of monetary policy. If the monetary authority is active, $|\phi_\pi| > 1$, the determinate solution is

$$\pi_t = \frac{1}{\phi_\pi} r_t. \tag{4.2}$$

Alternatively, if the monetary authority is passive, $|\phi_\pi| \leq 1$, the indeterminate solution is any process that takes the following form

$$\pi_t = \phi_\pi \pi_{t-1} - r_{t-1} + \eta_t. \tag{4.3}$$

The problem that a researcher faces when dealing with the indeterminate solution of a LRE model such as the one presented in (4.1) is the following. The equilibrium dynamics are uniquely determined if the Blanchard-Kahn condition is satisfied (Blanchard and Kahn, 1980b). The condition requires the number of expectational variables of the model to equal the number of its unstable roots. The endogenous variable of the univariate model in (4.1) is expectational and the dynamics properties of the model depends on the value assumed by ϕ_π. When $|\phi_\pi| > 1$, the model has a unique solution since it has a sufficient number of unstable roots to match the number of expectational variables. However, when $|\phi_\pi| \leq 1$, the model is indeterminate since it is missing one explosive root. The latter case constitutes a challenge because standard software packages do not deal with indeterminacy.

The approach in Bianchi and Nicolò (2017) proposes to augment the original model by appending an independent process which could be either stable or unstable. The key insight consists of choosing this auxiliary process in a way to deliver the correct solution. When the original model is determinate, the auxiliary process must be stationary so that also the augmented representation satisfies the Blanchard-Kahn condition. When the model is indeterminate, the additional process should however be explosive so that the Blanchard-Kahn condition is satisfied for the augmented system, even if it is not for the original model. In what follows, I apply this intuition to the example considered in this section and explain how to choose the auxiliary process.

Considering the univariate example in (4.1), the methodology of Bianchi and Nicolò (2017) proposes to solve the following augmented system of equations

$$\begin{cases} E_t(\pi_{t+1}) & = & \phi_\pi \pi_t - r_t, \\ \omega_t & = & \left(\frac{1}{\alpha}\right) \omega_{t-1} - \nu_t + \eta_t, \end{cases} \tag{4.4}$$

where ω_t is an auxiliary autoregressive process, $\alpha \in [0, 2]$, ν_t is a newly defined mean-zero sunspot shock with standard deviation σ_v and η_t still denotes the forecast errors, $\eta_t = \pi_t - E_{t-1}(\pi_t)$ as in the original model.[63]

Table 4.1 summarizes the intuition behind the approach. When the original LRE model in (4.1) is determinate, $|\phi_\pi| > 1$, the Blanchard-Kahn condition for the augmented representation in (4.4) is satisfied when $|1/\alpha| \leq 1$. Indeed, for $|\phi_\pi| > 1$ the original model has the same number of unstable roots as the number of expectational variables. The methodology thus suggests to append a stable autoregressive process and standard solution methods deliver the same solution for the endogenous variable π_t as in equation (4.2). Since the coefficient $|1/\alpha|$ is smaller than 1, the solution for the augmented representation also includes the

[63]The choice of parametrizing the auxiliary process with $1/\alpha$ instead of α induces a positive correlation between ϕ_π and α that facilitates the implementation of the method when estimating a model.

	Unstable Roots	B-K condition in augmented model (4.4)	Solution
Determinacy $\lvert\phi_\pi\rvert > 1$ in original model (4.1)			
$\frac{1}{\alpha}< 1$	1	Satisfied	$\left\{\pi_t = \frac{1}{\phi_\pi} r_t,\ \omega_t = \alpha\omega_{t-1} - \nu_t + \varepsilon_t\right\}$
$\frac{1}{\alpha}> 1$	2	Not satisfied	-
Indeterminacy $\lvert\phi_\pi\rvert \leq 1$ in original model (4.1)			
$\frac{1}{\alpha}< 1$	0	Not satisfied	-$\{\ \omega_t = 0\}$
$\frac{1}{\alpha}> 1$	1	Satisfied	$\left\{\begin{array}{ll} \pi_t = \phi_\pi \pi_{t-1} - r_{t-1} + \eta_t, & \\ \eta_t = v_t, & \omega_t = 0 \end{array}\right\}$

Table 4.1: Regions of the parameter space for which the Blanchard-Kahn condition in the augmented representation is satisfied, even when the original model is indeterminate.

autoregressive process ω_t. Importantly, its dynamics do not impact the endogenous variable y_t.

Considering the case of indeterminacy (i.e. $|\phi_\pi| \leq 1$), the original model has one expectational variable, but no unstable root, thus violating the Blanchard-Kahn condition. If the autoregressive process is explosive (i.e. $|1/\alpha| > 1$), the augmented representation satisfies the Blanchard-Kahn condition and delivers the same solution for π_t as in equation (4.3). Moreover, to guarantee boundedness, the solution imposes conditions such that ω_t is always equal to zero, and the solution for the endogenous variable, π_t, does not depend on the appended autoregressive process.

Summarizing, the choice of the additional parameter α should be made as follows. For values of $|\phi_\pi|$ outside the unit circle, the Blanchard-Kahn condition for the augmented representation is satisfied for values of $|1/\alpha|$ smaller than 1. Conversely, under indeterminacy (i.e. $|\phi_\pi| \leq 1$) the condition is satisfied when $|1/\alpha|$ is greater than 1. Also, note that under both determinacy and indeterminacy, the exact value of $1/\alpha$ is irrelevant for the law of motion of π_t. Under determinacy, the auxiliary process ω_t is stationary, but its evolution does not affect the law of motion of the model variables. Under indeterminacy, ω_t is always equal to zero. Hence, the introduction of the auxiliary processes does not affect the properties of the solution in either case. These processes only serve the purpose of providing the necessary explosive roots under indeterminacy and creating the mapping between the sunspot shocks and the expectational errors.

4.2 Implementation to Smets and Wouters (2007)

When adopting a univariate model such as in Section 4.1 or a small-scale model such as the NK model in LS, a researcher derives analytically the condition which partitions the parameter space into a determinate and indeterminate region. Also, she studies the dynamic properties of the model and determines the maximum degree of indeterminacy of the model. To implement the methodology developed in Bianchi and Nicolò (2017) to medium-scale models such as SW, a researcher faces the following technical complexities. It is not possible to derive analytically the partition of the parameter space, and the researcher does not know the exact properties of the determinacy region. Also, the adoption of a medium-scale model implies that a researcher does not know the degree of indeterminacy which characterizes the model.

To overcome these complexities, Bianchi and Nicolò (2017) indicate the following steps. First, the researcher should note that, for any model with p expectational variables, then the maximum degree of indeterminacy also corresponds to p. Defining $\{\eta_{i,t}\}_{i=1}^{p}$ to be the forecast errors associated with each expectational variable, the original LRE model should be augmented by appending up to p exogenous processes $\omega_{i,t} = \left(\frac{1}{\alpha_i}\right)\omega_{i,t-1} - \nu_{i,t} + \eta_{i,t}$ for $i = 1, .., p$. Second, the researcher cannot derive the partition of the parameter space analytically. For a given draw of the structural parameters of the model, the researcher would like to make draws of α_i smaller or greater than 1 with equal probabilities. Therefore, to implement this methodology to the model of SW, I assume a uniform distribution over the interval $[0.9, 1.1]$ as a prior distribution.[64] Third, while the newly defined shocks, $\{\nu_{i,t}\}_{i=1}^{p}$, are independent, they are potentially related to the structural shocks of the model. Hence, I assume a uniform distribution over the interval $[-1, 1]$ for the correlations between the newly defined shocks, $\{\nu_{i,t}\}_{i=1}^{p}$, and the seven structural shocks that impact the economy as described in Section 3.[65]

Following these steps, I find that the data favors a specification with one degree of indeterminacy. Hence, the augmented representation that I use to present the findings only includes one auxiliary process, ω_t. Also, the data indicates that the non-fundamental shock included in the augmented representation is the forecast error associated with the inflation rate $\eta_{\pi,t} \equiv \pi_t - E_{t-1}(\pi_t)$. In Section 5.1 I estimate the SW model augmented with the exogenous process $\omega_t = \left(\frac{1}{\alpha}\right)\omega_{t-1} - \nu_t + \eta_{\pi,t}$, where the newly defined sunspot shock, ν_t, could potentially be correlated with the seven structural shocks of the model. The estimation also shows that, according to the data, the correlation between the sunspot shock and the price mark-up shock is the only statistically significant.

5 Main Findings

I show that monetary policy was passive between 1955 and 1979, and active since 1984. As a result, the imposition of an active monetary policy as in SW delivers erroneous estimates

[64]Note that any symmetric interval around 1 also guarantees an equal probability of drawing α greater or smaller than 1. Alternatively, a researcher could assume a discrete distribution for which α could assume only two values (one inside the unit circle and one outside) with equal probabilities. However, this option is not implementable in standard software packages such as Dynare, since only continuous distributions are available as possible choice of prior distribution for the model parameters.

[65]From a technical perspective, the parameters which characterize the full set of indeterminate equilibria in LS relate to the covariances between the structural shocks of the model and a newly defined shock that their solution method introduces. As shown in Bianchi and Nicolò (2017), there is a unique mapping between their parametrization of the set of equilibria and the covariances which arise in this paper between the sunspot shock, ν_t, and the remaining structural shocks of the model. However, the additional parameters introduced in LS do not have a well-defined domain and the authors discipline the normal prior distributions for these parameters by centering them around the point estimates that minimize the distance between the impulse responses under determinacy and indeterminacy. On the contrary, the methodology of Bianchi and Nicolò (2017) that I adopt in this paper allows to deal with correlations, that are well-defined in the interval $[-1, 1]$ and for which a uniform distribution can be used as a prior.

of the structural parameters. I also analyze the conduct of U.S. monetary policy during the period between the collapse of the dot-com bubble and the Great Recession. The evidence of a passive monetary policy in a conventional three-equation NK model is instead ruled out when accounting for the rich dynamic and stochastic structure of the SW model.

I document the effects of a change in monetary policy on the dynamics of the economy and the transmission of structural shocks. When monetary policy is passive, the propagation of structural shocks is altered and more persistent due to the formation of self-fulfilling expectations. In this regime, a productivity shock still generates economic activity by decreasing the marginal cost incurred by the firms. However, the shock is also associated with the formation of persistent, inflationary expectations that more than offset the drop in marginal cost and finally result into self-fulfilling inflationary pressures.

Fundamental productivity and cost shocks were the primary drivers of the run-up in the inflation rate from the early 1960s to 1979. Positive technology shocks in the 1960s de-anchored inflation expectations from the central bank's long-run target and generated persistent inflationary pressures via self-fulfilling expectations. Mark-up shocks account for the sudden inflationary episodes related to the oil crisis during the 1970s, while they are not significant drivers of the rise in inflation during the 1960s. On the contrary, previous studies that impose an active monetary policy before 1979 exclude the role of self-fulfilling expectations for the transmission of structural shocks. The persistent rise in inflation from the early 1960s through the 1970s would be entirely and erroneously attributed to mark-up shocks. Moreover, the high volatility of inflation and output growth before 1979 was caused by fundamental disturbances, and non-fundamental sunspot shocks were not significant drivers of volatility between 1955 and 1979.

Finally, I revisit the question about the sources of the reduction in U.S. macroeconomic volatility between the early 1980s to 2007. Based on the SW model, I find that the reduction in macroeconomic uncertainty was a combination of both a change in monetary policy to a more active stance *and* a lower volatility of the shocks.

5.1 U.S. Monetary Policy in the Post-War Period

Section 5.1.1 provides evidence of a change in the conduct of monetary policy in the post-war period, from a passive stance before 1979 to an active inflation targeting since the early 1980s. This results has two implications. First, the assumption imposed in SW about an active monetary policy both before and after 1979 is rejected. Second, the findings in previous studies that adopted univariate or small-scale models (Clarida et al., 2000b, Lubik and Schorfheide, 2004) carry over to the SW model.

The section provides two additional findings. First, in Section 5.1.2 I show that, if a researcher assumes an active monetary policy before 1979, she would find erroneous estimates of the structural parameters, especially related to the persistence in inflation dynamics. The data would mistakenly indicate a higher degree of wage and inflation indexation as well as more persistence of the price mark-up shock.

Second, in Section 5.1.3 I shed light on the debate about whether the conduct of monetary policy after the dot-com bubble led to economic conditions that facilitated the occurrence of the Great Recession. I show that the results are susceptible to the modeling choice of the researcher due to the identification problem presented in Section 2.1. While the conventional three-equation NK model in LS rationalizes the data with a passive monetary policy, the adoption of the richer dynamic and stochastic structure in the SW model overturns this conclusion and indicates the conduct of an active monetary policy.

5.1.1 Changes in the Conduct of U.S. Monetary Policy

This section provides evidence of the change in the conduct of U.S. monetary policy in the post-war period. By considering the model and the data described in Section 3, I apply the methodology presented in Section 4 to estimate the SW model over three subsamples. The first period starts in 1955:4, which corresponds to one year after the end of the Korean War, and it ends in 1969:4, the date in which the chairmanship of William Martin terminates. The second sample considers the chairmanships of both Arthur Burns and William Miller, and it spans from 1970:1 until 1979:2. Finally, the beginning of the third period corresponds to 1984:1, in which Kim and Nelson (1999) initially identify a structural break in the U.S. business cycle, while the end is marked by the Great Recession in 2007:3.[66]

Appendix A reports the prior distributions for the structural parameters of the model and the exogenous processes that drive the dynamics of the economy. Relative to the prior distributions used in SW, the only difference relates to the Taylor rule coefficient associated with the response of the monetary authority to changes in the inflation rate. While SW specify a normal distribution truncated at 1, centered at 1.50 and with standard deviation 0.25, I consider a prior which assigns an approximately equal probability of observing indeterminacy as well as a unique solution. In particular, I set a flatter normal prior distribution centered at 1 and with standard deviation 0.35.

As discussed in Section 4.2, I estimate the model implementing the methodology developed in Bianchi and Nicolò (2017) and using Bayesian techniques. The data favors a specification with one degree of indeterminacy and in which the non-fundamental shock included in the augmented representation is the forecast error associated with the inflation rate $\eta_{\pi,t} \equiv \pi_t - E_{t-1}(\pi_t)$. Therefore, I estimate the SW model augmented it with the exogenous process $\omega_t = \left(\frac{1}{\alpha}\right) \omega_{t-1} - \nu_t + \eta_{\pi,t}$. For the parameter α, I assume a uniform prior distribution over the interval $[0.9, 1.1]$ and I also specify a uniform prior distribution over the interval $[0, 1]$ for the standard deviation of the sunspot shock, σ_ν.[67] Moreover, the data favors a specification in which the sunspot shock, ν_t, is correlated with price mark-up shock, while restricting the remaining correlations to 0. For the estimation, I therefore use a uniform distribution over the interval $[-1, 1]$ as the prior for the correlation between the price mark-up

[66] The findings in this section for the period prior to 1979:2 are quantitatively unchanged when considering a sample spanning from 1955:4 until 1979:2. However, studying the two samples separately is relevant to understand the connection between different steady state properties between the two periods and the exceptional growth in productivity until the early 1970s documented in Fernald (2014a).

[67] As shown in Table 5.3, the posterior distribution for the sunspot shock is not at the boundary but rather interior to the interval over which the uniform prior distribution.

shock and the sunspot shock.[68]

Table 5.1 reports the results of the estimation for each subsample. Relative to SW, the novelty is to relax the *a priori* assumption of equilibrium uniqueness. The method described in Section 4 allows to estimate the model over the entire parameter space. For each period, the Metropolis-Hastings algorithm finds two local maxima, one associated with the determinate solution and the other with the indeterminate representation. It is therefore possible to compute the corresponding marginal data density using the modified harmonic mean estimator proposed by Geweke (1999b) and the posterior model probabilities associated with each local maxima. Focusing on the first two samples that cover the period from 1955:4 to 1979:2, the data strongly favors the representation associated with indeterminacy, therefore rejecting the assumption of equilibrium uniqueness imposed in SW. On the contrary, the period subsequent to the Volcker disinflation is associated with a determinate, unique representation.

		Determinacy	Indeterminacy
Martin (55Q4 - 69Q4)	Log data density	-278.38	-272.50
	Posterior Model Prob (%)	0.0%	100.0%
Burns-Miller (70Q1 - 79Q2)	Log data density	-337.23	-319.29
	Posterior Model Prob (%)	0.0%	100.0%
Post-Volcker (84Q1 - 07Q3)	Log data density	-399.85	-406.88
	Posterior Model Prob (%)	100.0%	0.0%

Table 5.1: Log-data densities and the posterior model probabilities obtained for each sample period.

The evidence of a change in the monetary policy stance since 1984:1 is presented in Table 5.2, where the posterior distributions of the structural parameters in the three sub-periods are compared.[69] Considering the Taylor rule coefficient associated with the response of the

[68]The reader is referred to Section 4.2 for the technical details of the implementation of the methodology presented in Bianchi and Nicolò (2017) to medium-scale model of SW.

[69]I consider the posterior estimates to be unchanged when the posterior mean of a parameter estimated

monetary authority to changes in the inflation rate, r_π, it is clear that the monetary authority was passive prior to 1979, thus consistent with a weak response of the monetary authority to changes in the inflation rate. Table 5.2 also suggests that for the period subsequent to the Volcker disinflation, the monetary authority changed its stance and acted more aggressively to stabilize inflation, therefore ensuring equilibrium uniqueness.

Importantly, these results provide evidence that, even when accounting for the richer propagation mechanisms, equilibrium was indeterminate before 1979, and the findings of Clarida et al. (2000b) and LS among others carry over to a medium-scale model.

Table 5.2 also provides evidence in support of Fernald (2014a) who documents that the U.S. economy experienced a period of exceptional growth in productivity in the post-war period until the early 1970s. Both the trend growth rate of the economy and the (steady state) hours worked drop significantly in the period between 1970 until 1979 relative to the previous period. The posterior distributions also show that the post-Volcker period is characterized by a mildly higher degree of price stickiness, ξ_p, and a more persistent process of the price-markup shock measured by ρ_p in Table 5.3. This finding is supported by Galí and Gertler (1999), who provide evidence of an increased average price duration over this period due to the lower and more stable inflation rate. Also, the post-Volcker period is associated with a larger adjustment cost faced by the representative agent that chooses a higher degree of capital utilization rate.

Finally, the comparison in Table 5.3 of the properties of the exogenous processes between the period before and after 1979 provides an additional finding. In line with a large literature, the volatility of the shocks that drive fluctuations of the economy are significantly smaller starting from the mid 1980s (Stock and Watson, 2003, Primiceri, 2005, Sims and Zha, 2006). This result and the evidence of the change in the conduct of monetary policy are clearly linked to the discussion on the possible explanations for the sources of the reduction in U.S.

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in either of the two sample periods is within the 90% probability interval associated with the posterior distribution obtained in the alternative sample.

		1955:4-1969:4		1970:1-1979:2		1984:1-2007:3	
Coefficient	Description	Mean	[5 , 95]	Mean	[5 , 95]	Mean	[5 , 95]
ϕ	Adjustment cost	4.58	[2.68,6.38]	3.41	[2.01,4.64]	6.95	[5.20,8.73]
σ_c	IES	1.13	[0.85,1.40]	0.91	[0.67,1.15]	1.61	[1.38,1.84]
h	Habit Persistence	0.60	[0.48,0.73]	0.62	[0.49,0.76]	0.65	[0.57,0.73]
σ_l	Labor supply elasticity	1.98	[0.93,3.07]	1.29	[0.25,2.15]	2.29	[1.33,3.22]
ξ_w	Wage stickiness	0.73	[0.62,0.84]	0.70	[0.59,0.81]	0.68	[0.53,0.83]
ξ_p	Price Stickiness	0.59	[0.51,0.67]	0.58	[0.50,0.65]	0.75	[0.67,0.83]
ι_w	Wage Indexation	0.33	[0.14,0.53]	0.57	[0.36,0.78]	0.44	[0.20,0.68]
ι_p	Price Indexation	0.29	[0.12,0.45]	0.48	[0.25,0.73]	0.28	[0.10,0.44]
ψ	Capacity utiliz. elasticity	0.55	[0.36,0.75]	0.49	[0.26,0.72]	0.71	[0.57,0.86]
Φ	Share of fixed costs	1.59	[1.46,1.72]	1.34	[1.18,1.50]	1.60	[1.46,1.75]
α	Share of capital	0.24	[0.19,0.29]	0.18	[0.13,0.23]	0.23	[0.19,0.26]
$\bar{\pi}$	S.S. inflation rate (quart.)	0.62	[0.45,0.78]	0.62	[0.46,0.77]	0.68	[0.55,0.80]
$100(\beta^{-1}-1)$	Discount factor	0.17	[0.06,0.27]	0.21	[0.08,0.33]	0.13	[0.05,0.21]
$\bar{l}$	S.S. hours worked	1.36	[0.21,2.53]	-2.37	[-3.58,-1.05]	1.57	[0.36,2.80]
$\bar{\gamma}$	Trend growth rate (quart.)	0.47	[0.40,0.53]	0.36	[0.32,0.41]	0.45	[0.42,0.48]
r_π	Taylor rule inflation	0.64	[0.32,0.98]	0.75	[0.54,0.99]	1.80	[1.39,2.20]
r_y	Taylor rule output gap	0.13	[0.05,0.20]	0.16	[0.09,0.23]	0.09	[0.03,0.14]
$r_{\Delta y}$	Taylor rule Δ(output gap)	0.11	[0.07,0.15]	0.18	[0.12,0.24]	0.15	[0.10,0.19]
ρ	Taylor rule smoothing	0.87	[0.81,0.95]	0.73	[0.60,0.86]	0.84	[0.80,0.88]

Table 5.2: Posterior estimates comparison of structural parameters under indeterminacy for the pre-Volcker and under determinacy for the post-Volcker period.

macroeconomic volatility from the early 1980s to 2007. In Section 5.4, I show that, according to the SW model, both the change in the monetary policy stance *and* the lower size of the shocks explain this empirical observation for U.S. macro data.

		1955:4-1969:4		1970:1-1979:2		1984:1-2007:3	
Coefficient	Description	Mean	[5 , 95]	Mean	[5 , 95]	Mean	[5 , 95]
σ_a	Technology shock	0.52	[0.44,0.61]	0.56	[0.45,0.67]	0.36	[0.31,0.40]
σ_b	Risk premium shock	0.19	[0.11,0.27]	0.17	[0.10,0.23]	0.18	[0.14,0.22]
σ_g	Government sp. shock	0.51	[0.43,0.59]	0.55	[0.44,0.65]	0.41	[0.36,0.46]
σ_I	Investment-specific shock	0.60	[0.42,0.77]	0.38	[0.23,0.53]	0.35	[0.28,0.43]
σ_r	Monetary policy shock	0.11	[0.09,0.12]	0.22	[0.18,0.26]	0.12	[0.10,0.14]
σ_p	Price mark-up shock	0.24	[0.20,0.29]	0.31	[0.24,0.39]	0.09	[0.07,0.11]
σ_w	Wage mark-up shock	0.24	[0.19,0.28]	0.31	[0.23,0.38]	0.31	[0.24,0.37]
σ_ν	Sunspot shock	0.14	[0.07,0.21]	0.19	[0.06,0.33]	-	-
ρ_a	Persistence technology	0.95	[0.92,0.99]	0.73	[0.60,0.87]	0.92	[0.87,0.97]
ρ_b	Persistence risk premium	0.59	[0.35,0.84]	0.77	[0.62,0.92]	0.20	[0.05,0.35]
ρ_g	Persistence government sp.	0.86	[0.78,0.94]	0.85	[0.77,0.94]	0.96	[0.94,0.98]
ρ_I	Persistence investment-specific	0.50	[0.30,0.70]	0.65	[0.47,0.84]	0.64	[0.52,0.76]
ρ_r	Persistence monetary policy	0.50	[0.31,0.68]	0.32	[0.11,0.51]	0.37	[0.21,0.52]
ρ_p	Persistence price mark-up	0.24	[0.04,0.43]	0.39	[0.11,0.65]	0.83	[0.72,0.95]
ρ_w	Persistence wage mark-up	0.63	[0.36,0.91]	0.42	[0.14,0.68]	0.81	[0.66,0.95]
μ_p	MA price mark-up	0.64	[0.43,0.85]	0.70	[0.45,0.95]	0.66	[0.48,0.84]
μ_w	MA wage mark-up	0.50	[0.27,0.75]	0.56	[0.26,0.88]	0.61	[0.38,0.82]
ρ_{ga}	$Cov(\sigma_a, \sigma_g)$	0.59	[0.39,0.78]	0.62	[0.40,0.84]	0.40	[0.22,0.57]
$\rho_{\nu p}$	$Corr(\sigma_\nu, \sigma_p)$	0.92	[0.82,0.99]	0.69	[0.37,0.99]	-	-

Table 5.3: Posterior estimates comparison of the parameters associated with the exogenous processes under indeterminacy for the pre-Volcker and under determinacy for the post-Volcker period.

5.1.2 The Impact of the SW Restriction

This subsection studies the implications of the *a priori* restriction about equilibrium uniqueness imposed in SW for the study of U.S. business cycle fluctuations. As shown in Table 5.1, the assumption is validated by the data exclusively for the post-Volcker period. On the contrary, the restriction is rejected when considering the sample prior to 1979. Table 5.4 reports the posterior distribution of the structural parameters estimated for each of the two local maxima found by the Metropolis-Hastings algorithm for the first sample period (1955:4-1969:4). The table allows for a comparison with the estimation results that would be obtained by imposing the same *a priori* assumption as in SW.[70] While most of the estimates are unchanged, relaxing the restriction implies that the Taylor rule coefficient on inflation is estimated to be associated with a weak response of the monetary authority, therefore rejecting the assumption imposed in SW. As shown in Section 5.2 and 5.3, this finding has crucial implications for the propagation of the shocks and to explain U.S. business cycle fluctuations.

The comparison of the posterior estimates also highlights a higher degree of both the wage and inflation indexation, as well as more persistence of the price mark-up shock. This finding is in line with the intuition provided in Section 2.1. A characteristic feature of indeterminate models is their richer endogenous persistence. Hence, when imposing the assumption of an active monetary policy, the model incurs a difficulty in matching the observed persistence in the data and mistakenly suggests a higher persistence than in the representation favored by the data.

[70]Similar differences arise when using the second subsample (1970:1-1979:2) to study how the imposition of the assumption in SW would impact the results.

Period: 1955:4-1969:4		Indeterminacy		Determinacy	
Coefficient	Description	Mean	[5 , 95]	Mean	[5 , 95]
ϕ	Adjustment cost	4.58	[2.68,6.38]	4.95	[3.11,6.74]
σ_c	IES	1.13	[0.85,1.40]	1.18	[0.81,1.55]
h	Habit Persistence	0.60	[0.48,0.73]	0.61	[0.44,0.79]
σ_l	Labor supply elasticity	1.98	[0.93,3.07]	1.43	[0.35,2.34]
ξ_w	Wage stickiness	0.73	[0.62,0.84]	0.76	[0.67,0.84]
ξ_p	Price Stickiness	0.59	[0.51,0.67]	0.62	[0.50,0.73]
ι_w	Wage Indexation	0.33	[0.14,0.53]	0.43	[0.20,0.65]
ι_p	Price Indexation	0.29	[0.12,0.45]	0.39	[0.12,0.68]
ψ	Capacity utiliz. elasticity	0.55	[0.36,0.75]	0.46	[0.25,0.66]
Φ	Share of fixed costs	1.59	[1.46,1.72]	1.62	[1.46,1.78]
α	Share of capital	0.24	[0.19,0.29]	0.24	[0.20,0.29]
$\bar{\pi}$	S.S. inflation rate (quart.)	0.62	[0.45,0.78]	0.62	[0.48,0.75]
$100(\beta^{-1}-1)$	Discount factor	0.17	[0.06,0.27]	0.18	[0.06,0.29]
$\bar{l}$	S.S. hours worked	1.36	[0.21,2.53]	2.03	[0.72,3.47]
$\bar{\gamma}$	Trend growth rate (quart.)	0.47	[0.40,0.53]	0.47	[0.29,0.60]
r_π	Taylor rule inflation	0.64	[0.32,0.98]	1.37	[0.99,1.71]
r_y	Taylor rule output gap	0.13	[0.05,0.20]	0.14	[0.06,0.23]
$r_{\Delta y}$	Taylor rule Δ(output gap)	0.11	[0.07,0.15]	0.12	[0.08,0.17]
ρ	Taylor rule smoothing	0.87	[0.81,0.95]	0.87	[0.82,0.92]

Table 5.4: Posterior estimates comparison of structural parameters for the pre-Volcker period under indeterminacy and determinacy.

Period: 1955:4-1969:4		Indeterminacy		Determinacy	
Coefficient	Description	Mean	[5 , 95]	Mean	[5 , 95]
σ_a	Technology shock	0.52	[0.44,0.61]	0.53	[0.43,0.62]
σ_b	Risk premium shock	0.19	[0.11,0.27]	0.17	[0.05,0.29]
σ_g	Government sp. shock	0.51	[0.43,0.59]	0.50	[0.42,0.58]
σ_I	Investment-specific shock	0.60	[0.42,0.77]	0.58	[0.40,0.75]
σ_r	Monetary policy shock	0.11	[0.09,0.12]	0.11	[0.09,0.14]
σ_p	Price mark-up shock	0.24	[0.20,0.29]	0.22	[0.15,0.29]
σ_w	Wage mark-up shock	0.24	[0.19,0.28]	0.24	[0.19,0.30]
σ_ν	Sunspot shock	0.14	[0.07,0.21]	-	-
ρ_a	Persistence technology	0.95	[0.92,0.99]	0.93	[0.85,0.99]
ρ_b	Persistence risk premium	0.59	[0.35,0.84]	0.64	[0.17,0.98]
ρ_g	Persistence government sp.	0.86	[0.78,0.94]	0.85	[0.74,0.96]
ρ_I	Persistence investment-specific	0.50	[0.30,0.70]	0.53	[0.33,0.74]
ρ_r	Persistence monetary policy	0.50	[0.31,0.68]	0.44	[0.27,0.60]
ρ_p	Persistence price mark-up	0.24	[0.04,0.43]	0.64	[0.22,0.98]
ρ_w	Persistence wage mark-up	0.63	[0.36,0.91]	0.52	[0.22,0.81]
μ_p	MA price mark-up	0.64	[0.43,0.85]	0.75	[0.46,0.99]
μ_w	MA wage mark-up	0.50	[0.27,0.75]	0.47	[0.19,0.75]
ρ_{ga}	$Cov(\sigma_a, \sigma_g)$	0.59	[0.39,0.78]	0.56	[0.37,0.76]
$\rho_{\nu p}$	$Corr(\sigma_\nu, \sigma_p)$	0.92	[0.82,0.99]	-	-

Table 5.5: Posterior estimates comparison of the parameters associated with the exogenous processes for the pre-Volcker period under indeterminacy and determinacy.

5.1.3 The Federal Reserve Leading to the Great Recession?

The framework considered in this paper also allows to shed light on the recent debate on the conduct of U.S. monetary policy during the period between the collapse of the dot-com bubble and the Great Recession. On the one hand, Taylor (2012) considers the headline consumer price index (CPI) to measure the inflation rate and suggests that, by keeping the federal fund rate too low relative to a conventional Taylor rule since the 2001, the Fed generated economic conditions which led to the Great Recession. On the other hand, Bernanke (2015) constructs a measure of inflation using the core personal consumption expenditure deflator (PCE) and finds that the Fed reacted as prescribed by a conventional Taylor rule to changes in the inflation rate.

Doko Tchatoka et al. (2017) assess the performance of the Fed by using a structural approach and in particular a conventional three-equation NK model in the spirit of LS. The authors find that monetary policy was active only when the inflation rate is measured with core PCE. However, when the same analysis is conducted using headline CPI to measure inflation, the evidence of equilibrium indeterminacy cannot excluded.

In this section, I argue that, after accounting for the richer dynamic and stochastic structure of the SW model, the evidence of a passive monetary is overturned. As in Doko Tchatoka et al. (2017), I focus on the period between the 2001 slump and the onset of the Great Recession (2002:1-2007:3) and I use the GDP deflator to measure inflation as in SW. However, I address the question about the conduct of U.S. monetary policy by estimating both the small-scale model in LS and the medium-scale model of SW.[71]

Table 5.6 reports the (log) data densities and the corresponding marginal data densities for the determinate and indeterminate representations using two alternative models. The first row is in line with the result of Taylor (2012) and Doko Tchatoka et al. (2017). By estimating the small-scale model of LS, the data provides evidence of indeterminacy with a posterior probability of 78.8%. Nevertheless, the conclusion is reversed once richer and more relevant propagation mechanisms and structural shocks are included. According to the SW model, monetary policy was active and consistent with a determinate equilibrium. Finally, Table 5.6 provides an empirical example of the identification problem described in Section 2.1 for which missing propagation mechanisms could be misinterpreted as evidence of indeterminacy.

[71]Regarding the estimation of the SW model, I use the same model as in the Section 3 and I restrict the data to consider the sample under study. The LS model is described in Section 5.4 and I use the same data as for SW for the observables of the model: the output gap, the inflation rate and the nominal interest rate. As a note, since the SW model assumes the uncorrelation of the structural shocks (except the productivity and government spending shocks), I also assume that the structural shocks of the LS model (mark-up, demand and monetary policy shocks) are uncorrelated.

Sample: 2002:1-2007:3		Determinacy	Indeterminacy
LS model	Log data density	-20.48	-19.16
	Posterior Model Prob (%)	21.2%	78.8%
SW model	Log data density	-122.20	-125.76
	Posterior Model Prob (%)	97.2%	2.8%

Table 5.6: Log-data densities and the posterior model probabilities for the LS model and the SW model using the sample period 2002:1-2007:3.

5.2 Monetary Policy, Expectations and the Propagation Mechanism

In this section, I focus on the implications that the observed change in the stance of monetary policy has on the transmission of the structural shocks of the SW model. In particular, I study the propagation of three shocks that, as highlighted in Section 5.3, explain most of U.S. business cycle fluctuations in the period prior to 1979: productivity, risk-premium and monetary policy shocks.[72]

Productivity Shock The impact of a productivity shock has implications that differ depending on the conduct of U.S. monetary policy. The four panels on the right of Figure 5.1 show the transmission of a (one standard deviation) productivity shock in the post-Volcker period on the output gap, the inflation rate, the nominal interest rate and the marginal cost incurred by firms.[73] The shock generates economic activity and deflationary pressures due to a drop in marginal cost. Under the active inflation targeting, the monetary authority responds by lowering the policy rate by more than one-for-one. Conversely, the four panels on the left are associated with the passive monetary policy of the Burns and

[72]Regarding the remaining shocks, either the propagation mechanism is mostly independent of the conduct of monetary policy or the shocks do not play a major role for U.S. business cycles.

[73]The size of the shock depends on the standard deviation estimated in each of the two samples. As found in Table 5.3, the size of the shock in the two samples before 1979 is larger that the standard deviation estimated for the post-Volcker period.

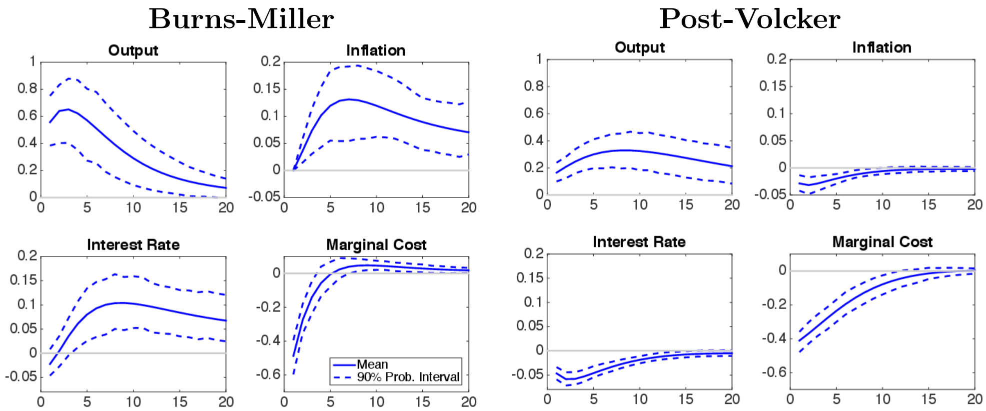

Figure 5.1: Mean impulse responses to a productivity shock are denoted by solid lines, while dashed lines represent the associated 90% probability intervals.

Miller chairmanship. The shock still results into a drop in marginal costs and an economic expansion. However, the productivity shock also generates inflationary expectations that are not suppressed by the passive monetary authority and more than compensate the drop in marginal cost. This mechanism thus results into a self-fulfilling rise of the inflation rate.[74] The corresponding increase in the nominal interest rate is gradual and not aggressive enough to stabilize the inflation rate, therefore allowing for persistent effects on the economy.

Risk-Premium Shock The risk-premium shock represents a wedge between the policy rate set by the central bank and the return that households receive to hold their assets. As Figure 5.2 suggests, the shock has similar effects on the real economy regardless of the conduct of monetary policy. A (one standard deviation) negative shock increases consumption since the required rate of return on assets is lower. Also, the decrease in the cost of capital further stimulates economic activity due to larger investments by firms. However, the inflation response to the risk-premium shock depends on the conduct of monetary policy. When monetary policy is active, firms face a higher marginal cost that maps into inflationary pressures. When monetary policy is passive, agents observe a rise in the real interest rate

[74] As discussed in Section 4.2, the data favors a specification which includes the forecast error associated with the inflation rate, $\eta_{\pi,t}$, as a non-fundamental shock. This implies that the inflation rate is predetermined as a function of the previous period's conditional expectation, $\pi_t = E_{t-1}(\pi_t)+\eta_{\pi,t}$. Equivalently, the inflation rate is not affected on impact.

and form self-fulfilling deflationary expectations due to the convergence of the economy to its long-run steady state. In this case, the risk-premium shock therefore dampens the inflation rate of the economy.

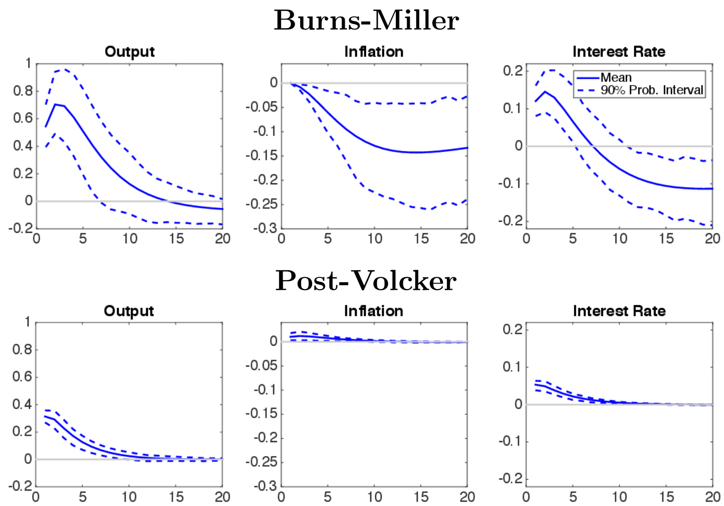

Figure 5.2: Mean impulse responses to a risk-premium shock are denoted by solid lines, while dashed lines represent the associated 90% probability intervals.

Monetary Policy Shock The bottom three panels of Figure 5.3 describe the predictions of a contractionary monetary policy shock under the active regime of the post-Volcker period. Output and inflation drop and revert to the steady state of the economy. When monetary policy is indeterminate, the responses to a contractionary monetary policy shock are reported in the top three panels. Economic activity is depressed. However, in line with the empirical findings of LS, the unexpected tightening of monetary policy is associated with a persistent inflationary effect. Agents form inflationary expectations due to the convergence of the economy back to its long-run. These expectations are then self-fulfilled and the contractionary monetary policy shock results into a persistent inflationary effect. Therefore, Figure 5.3 highlights the differences in the impact and the transmission of structural shocks such as an unexpected monetary policy tightening.

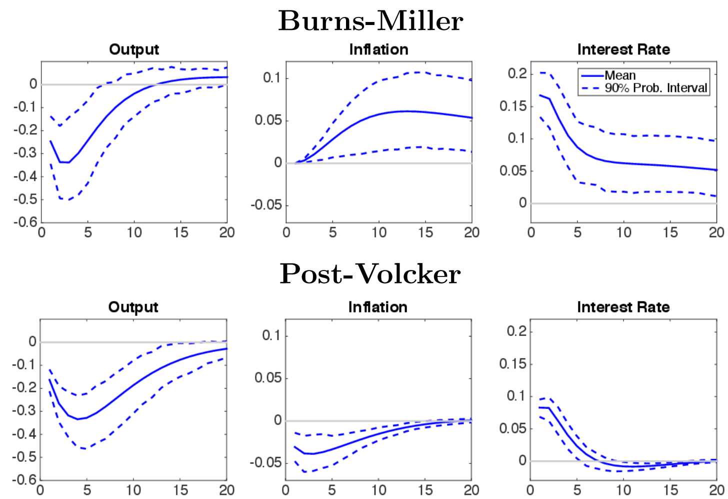

Figure 5.3: Mean impulse responses to a monetary policy shock are denoted by solid lines, while dashed lines represent the associated 90% probability intervals.

5.3 The History of U.S. Business Cycles

The interpretation of U.S. business cycle fluctuations relies on the conduct of monetary policy. I find that for the period prior to 1979, persistent technology shocks explain the upward trend in the inflation rate observed since the early 1960s. This result is in line with the observation of Fernald (2014a) according to which the U.S. economy experienced a period of exceptional growth in productivity since World War II until the oil crisis in 1973. Mark-up shocks account for the sudden inflationary episodes related to the oil price shocks during the early 1970s, but do not explain the persistent rise in the inflation rate. Sunspot shocks play a minor role to explain the high macroeconomic volatility observed before 1979. These findings indicate that the strong evidence of a passive monetary policy before 1979 lies in the persistence of the distinctive transmission mechanism of structural shocks rather than in the quantitative relevance of non-fundamental disturbances. Regarding the period after the Volcker disinflation, the results are in line with previous findings in the literature. The recessionary episodes in the early 1990s and the burst of dot com bubble are mostly explained by negative demand shocks, and mark-up shocks kept the inflation rate subdued relative to its target level during the 1990s.

5.3.1 Martin and the Post-Korean War Period

I first focus on the post-Korean war period that starts in 1955 and during which William Martin has been the chairman of the Fed until 1969. Figure L.1 plots the historical decomposition of the deviations of the inflation rate from its long-run for two alternative specifications.[75]

The top panel of Figure 5.4 plots the decomposition under indeterminacy that is favored by the data.[76] While sunspot shocks could have potentially contributed for the model to better match the high volatility in the inflation rate, the historical decomposition indicates that they played no quantitative role. The rise in inflation since the early 1960s is associated with a sequence of productivity shocks. In line with the analysis in Section 5.2 on the persistent inflationary effects induced by positive productivity shocks, the top panel indicates that the impact of each shock cumulated over time and explained the upward trend in the inflation rate.[77] As discussed below, this result is supported by the empirical evidence presented in Fernald (2014a) among others.

The bottom panel reports the decomposition that results by imposing equilibrium uniqueness as conducted in SW.[78] A comparison with the top panel also suggests that the assumption substantially affects the interpretation of the data. The upward trend in the inflation rate during the 1960s is erroneously attributed to mark-up shocks. However, the results in Section 5.1 reject the assumption, thus indicating that the correct interpretation relies on

[75]The historical decomposition of output growth for the two sample periods prior to 1979 is provided in Appendix B and shows minor differences between the determinate and the indeterminate representation.

[76]To conduct the historical decompositions, I use the posterior means estimated for the pre-Volcker period for each of the two local maxima found during the estimation and that are reported in Table 5.2 and 5.3. Also, to simplify the analysis, I group the exogenous spending shock, the investment-specific shock and the risk-premium shock as "demand" shocks. Similarly, price and wage mark-up shocks are grouped as "mark-up" shocks.

[77]Monetary policy shocks had a minor impact that resulted into mildly deflationary pressures during the early 1960s. It is useful to recall that the historical decomposition cumulates the effect of a given shock on the inflation rate until a given date. Given the persistence of the monetary policy shocks under indeterminacy as described in Section 5.2, a monetary policy shock can be identified as the change in the contribution to explain the dynamics.

[78]Minor differences in the historical decomposition at a given year across the two panels are explained by differences in the contribution of the initial conditions for each representation.

the top panel.

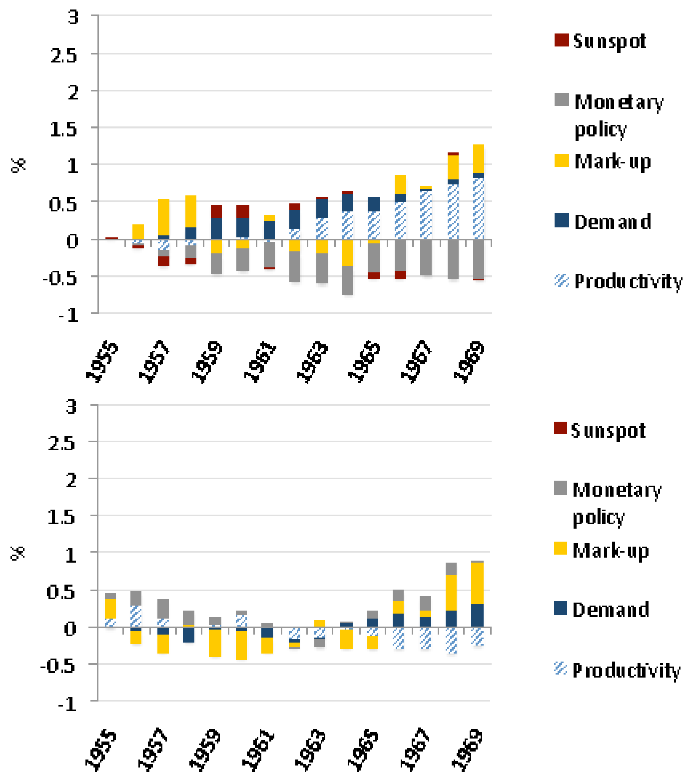

Figure 5.4: Sample 1955-1969. Historical decomposition of the inflation rate under indeterminacy (top) and determinacy (bottom) at quarterly rates.

The evidence that the U.S. economy experienced exceptional growth in productivity prior to the early 1970s is documented by Fernald (2014a) among others[79]. This literature points to a wave of technological innovations as the source of a rise in growth of productivity and therefore economic activity. In particular, when considering the quarterly time series for Total Factor Productivity produced by Fernald (2014b), the resulting (standardized) series is plotted in Figure 5.5 together with the smoothed productivity shocks estimated using the SW model. The comparison indicates that the estimation of the SW model successfully identifies the sequence of positive productivity shocks that the U.S. economy experienced starting from the early 1960s.[80] Importantly, in a passive monetary policy regime, productivity shocks

[79]Other work that supports this view is provided by Fernald (2014b), Gordon (2000), Davig and Wright (2000) and Field (2003).

[80]The correlation between the two sequences of productivity shocks is 0.74, suggesting that the model does

generate persistent inflationary expectations that are consistent with the observed upward trend in inflation.

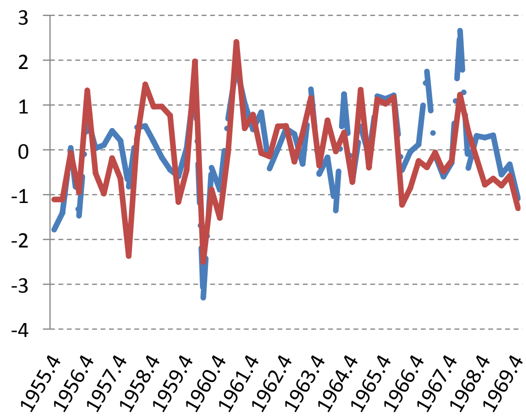

Figure 5.5: Sample 1955:4-1969:4. Quarterly (standardized) series of Total Factor Productivity from Fernald (2014a, solid line) and of the smoothed productivity shocks from SW model (dashed line) in percentages at quarterly rates.

5.3.2 The Burns and Miller Chairmanships

The second period begins with the chairmanship of Arthur Burns in 1970 and ends in 1979 with the conclusion of the chairmanship of William Miller. Figure 5.6 presents the historical decomposition of the inflation rate over this sample period according to alternative monetary policy regimes. The top panel presents the decomposition associated with indeterminacy as supported by the data, while the bottom panel is obtained by imposing the assumption that monetary policy successfully suppressed self-fulfilling expectations. As explained in Section 5.2, the conduct of a passive monetary policy is such that positive risk-premium shocks have a contractionary effect on the economy but also lead agents to form inflationary expectations that are self-fulfilling and persistent.[81] Hence, a combination of demand shocks and positive

remarkably well in extracting productivity shocks that are in line with Fernald (2014b).

[81] The drop in the inflation rate between 1969 in Figure 5.4 and 1970 in Figure 5.6 is due to a relatively mild recession that coincided with an attempt of the U.S. government to start closing the budget deficits of the Vietnam War. Hence, the decomposition attributes most of the drop to the initial condition since the economy is not at its steady state. However, as mentioned in Section 5.1.1, it is relevant to account for the two samples separately since the balanced growth path of the economy differs substantially.

productivity shocks sustained the high inflation observed in the late 1970s, while the spike in 1979 is also attributed to mark-up shocks.[82] Even for this sample period, sunspot shocks have no quantitative relevance for U.S. business cycles. Conversely, the bottom panel shows that the assumption of an active monetary policy mistakenly attributes the fluctuations in the inflation rate exclusively to mark-up shocks.

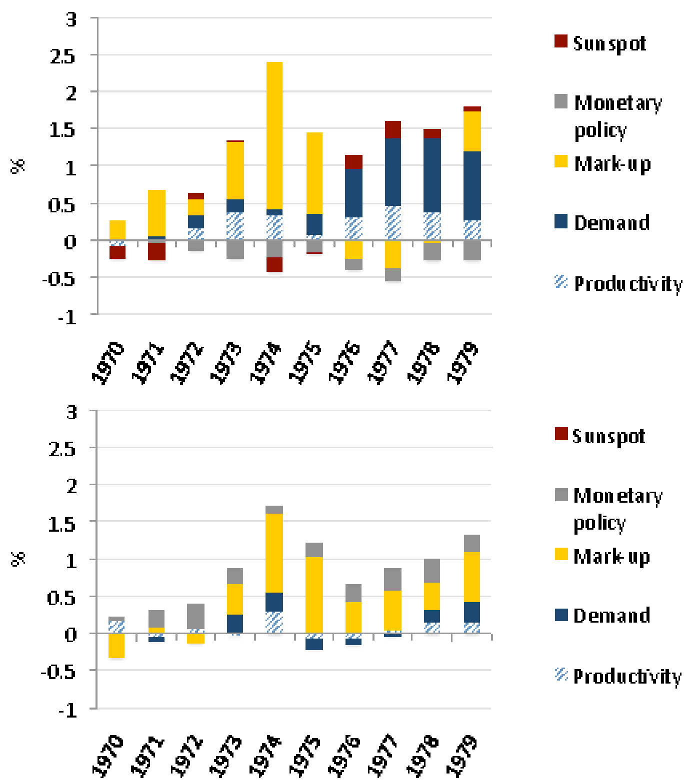

Figure 5.6: Sample 1970-1979. Historical decomposition of the deviation of the inflation rate from its steady state under indeterminacy (top) and determinacy (bottom) at quarterly rates.

5.3.3 The Post-Volcker Period

Finally, I focus on the post-Volcker period and Figure 5.7 reports the historical decomposition for the output gap (top panel) and the inflation rate (bottom panel). The decomposition is conducted under an active monetary policy as found in Section 5.1 for this sample period.

[82]While Figure 5.6 generally refers to demand shocks, the break down for each demand shock shows that the contribution of the risk-premium shock is the most relevant as opposed to the government spending or investment-specific shocks.

The results are in line with those in SW. The economic contractions and below-target inflation rate of the early 1990s and 2000s are mostly explained by negative demand shocks, while mark-up shocks maintained the inflation rate subdued during the 1990s.

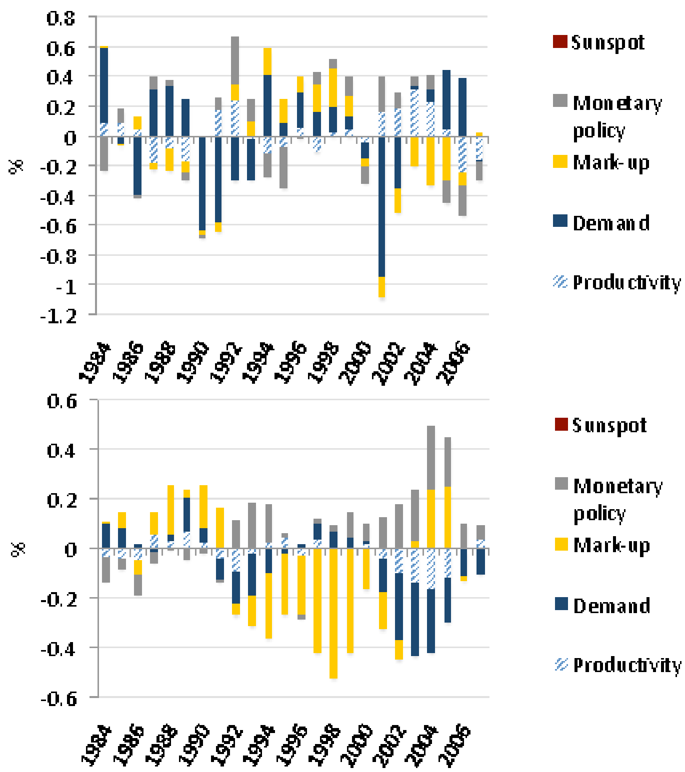

Figure 5.7: Post-Volcker sample. Historical decomposition of output gap (top) and inflation rate (bottom) at quarterly rates.

5.4 What Changed in the Early 1980s?

The work of Kim and Nelson (1999) and McConnell and Perez-Quiros (2000) first documented the lower volatility of U.S. real GDP since the early-1980s. Extensive research has been conducted to provide explanations for the reduction in U.S. macroeconomic volatility. Using the SW model, I investigate the validity of two prominent theories that have been advocated to explain this empirical fact. The work by Sims and Zha (2006) suggests that the behavior of the data changed due to a decrease in the variance of the structural shocks since the Volcker disinflation. Primiceri (2005) finds some evidence that policy also changed, but the role played by structural disturbances is more relevant. According to this strand of the literature, the reduction in volatility of U.S. macroeconomic data is not related to monetary policy and it can therefore be considered as "good luck".

An alternative theory has been supported by the work of Clarida et al. (2000b) and LS among others who find evidence of an active inflation targeting since the Volcker disinflation. The reduction in volatility can therefore be attributed to the "good policy" of the monetary authority.

The results in Section 5.1 support both theories. The comparison of the posterior estimates of the structural parameters in Table 5.2 indicates that the conduct of monetary policy changed toward a more aggressive stance in the post-Volcker period. Moreover, the estimates of the volatility of the shocks driving the economy dropped significantly since the early 1980s (Table 5.3). In this section, I show that, according to the SW model, *both* theories contribute to explain the reduction in U.S. macroeconomic volatility.

To provide an intuition for the approach that I adopt using the SW model, I first consider a conventional three-equation NK model such as in LS. The model is described by a dynamic IS curve

$$y_t = E_t\left(y_{t+1}\right) - \tau\left(R_t - E_t\left(\pi_{t+1}\right)\right) + z_{d,t},$$

a NK Phillips Curve

$$\pi_t = E_t\left(\pi_{t+1}\right) + \kappa\left(y_t - z_{s,t}\right),$$

and a monetary policy reaction function

$$R_t = \phi_\pi \pi_t + \varepsilon_{R,t},$$

where y_t represents the deviation of output from its trend and the demand shock, $z_{d,t}$, and supply shock, $z_{s,t}$, are autoregressive processes of the form

$$z_{d,t} = \rho_d z_{d,t-1} + \varepsilon_{d,t} \qquad\qquad z_{s,t} = \rho_s z_{s,t-1} + \varepsilon_{s,t}.$$

	Log data density	Posterior model prob	
Policy and shocks	-986.85	100%	100%
Shocks only	-994.51	0%	-
Policy only	-1021.08	-	0%

Table 5.7: Marginal data densities for the three alternative specifications and the pairwise posterior model probabilities relative to Model 1, Policy and Shocks.

To test alternative theories, I estimate different model specifications by imposing restrictions on sets of parameters and volatilities. In Model 1, "Policy and Shocks", I constrain the private sector parameters, $\{\tau, \kappa\}$, to be the same across the period from 1955 to 1979 and the period from 1984 to 2007. I also allow the policy parameter, ϕ_π, the variances of the shocks, $\{\sigma_d, \sigma_s\}$, and the autoregressive coefficients, $\{\rho_d, \rho_s\}$, to vary across the two periods. This specification considers a combination of "good luck" and "good policy" to explain the data. Relative to Model 1, I then consider Model 2, "Shocks only", by further restricting the policy parameter, ϕ_π, thus considering the "good luck" view. Conversely, consistent with the "good policy" theory, Model 3, "Policy only", allows for the policy parameter to vary across sub-periods, while constraining all the other structural parameters and variances to be constant. Following the intuition provided with the conventional three-equation NK model, I apply the same approach to estimate alternative model specifications of the SW model and test for the validity of the "good luck" and/or "good policy" theory in the data.

Table 5.7 reports the marginal data densities obtained for the estimation of each of the three models. The posterior model probabilities are computed as pairwise comparisons relative to Model 1, Policy and Shocks, and indicate that, based on the SW model, a combination of both good luck and good policy is the explanation for the observed reduction in the volatility since the mid-1980s. Also, in line with the findings of Primiceri (2005), Model 2, Shocks only, has a more relevant role rather than the theory based exclusively on the change of monetary policy to an active stance, Model 3 Policy only.

Focusing on the estimation of Model 1, Policy and Shocks, Table 5.8 reports the posterior estimates of the constrained parameters across the two sub-samples. As expected, the posterior distribution of the model parameters are in line with those estimated for the two samples separately and reported in Section 5.1.

Coefficient	Description	Mean	[5 , 95]
ϕ	Adjustment cost	6.59	[4.75,8.48]
σ_c	IES	1.44	[1.25,1.63]
h	Habit Persistence	0.62	[0.51,0.73]
σ_l	Labor supply elasticity	2.14	[1.37,2.91]
ξ_w	Wage stickiness	0.84	[0.79,0.89]
ξ_p	Price Stickiness	0.77	[0.70,0.83]
ι_w	Wage Indexation	0.40	[0.24,0.56]
ι_p	Price Indexation	0.18	[0.06,0.30]
ψ	Capacity utiliz. elasticity	0.68	[0.55,0.81]
Φ	Share of fixed costs	1.63	[1.50,1.75]
α	Share of capital	0.22	[0.19,0.25]
$\bar{\pi}$	S.S. inflation rate (quart.)	0.60	[0.48,0.71]
$100(\beta^{-1}-1)$	Discount factor	0.10	[0.04,0.16]
$\bar{l}$	S.S. hours worked	0.67	[-0.29,1.55]
$\bar{\gamma}$	Trend growth rate (quart.)	0.42	[0.39,0.45]

Table 5.8: Posterior estimates of the constrained structural parameters.

Table 5.9 and 5.10 highlight substantial differences in the posterior estimates for the policy parameters and the exogenous processes between the two periods. Monetary policy acted more systematically to stabilize inflation. Consistent with the findings of Sims and Zha (2006) among others, the magnitude of the volatility of the shocks is significantly reduced in the post-Volcker period.

		Pre-Volcker (55:4 - 79:2)		Post-Volcker (84:1 - 07:3)	
Coefficient	Description	Mean	[5 , 95]	Mean	[5 , 95]
r_π	Taylor rule inflation	0.85	[0.73,0.97]	1.80	[1.37,2.18]
r_y	Taylor rule output gap	0.15	[0.09,0.21]	0.05	[0.02,0.10]
$r_{\Delta y}$	Taylor rule Δ(output gap)	0.17	[0.12,0.22]	0.17	[0.12,0.21]
ρ	Taylor rule smoothing	0.86	[0.80,0.91]	0.84	[0.80,0.88]

Table 5.9: Posterior estimates of policy parameters for the pre- and post-Volcker period.

		Pre-Volcker (55:4 - 79:2)		Post-Volcker (84:1 - 07:3)	
Coefficient	Description	Mean	[5 , 95]	Mean	[5 , 95]
σ_a	Technology shock	0.54	[0.47,0.60]	0.36	[0.31,0.40]
σ_b	Risk premium shock	0.14	[0.08,0.19]	0.15	[0.08,0.21]
σ_g	Government sp. shock	0.53	[0.46,0.60]	0.41	[0.36,0.46]
σ_I	Investment-specific shock	0.45	[0.35,0.56]	0.30	[0.23,0.37]
σ_r	Monetary policy shock	0.17	[0.15,0.20]	0.12	[0.10,0.14]
σ_p	Price mark-up shock	0.30	[0.25,0.34]	0.09	[0.07,0.11]
σ_w	Wage mark-up shock	0.26	[0.22,0.30]	0.31	[0.25,0.37]
σ_ν	Sunspot shock	0.06	[0.01,0.11]	-	-
ρ_a	Persistence technology	0.97	[0.96,0.98]	0.93	[0.89,0.96]
ρ_b	Persistence risk premium	0.75	[0.55,0.93]	0.38	[0.05,0.71]
ρ_g	Persistence government sp.	0.90	[0.86,0.95]	0.97	[0.95,0.98]
ρ_I	Persistence investment-specific	0.68	[0.55,0.81]	0.74	[0.62,0.86]
ρ_r	Persistence monetary policy	0.35	[0.20,0.51]	0.32	[0.19,0.45]
ρ_p	Persistence price mark-up	0.24	[0.04,0.42]	0.82	[0.73,0.92]
ρ_w	Persistence wage mark-up	0.34	[0.12,0.55]	0.69	[0.50,0.88]
μ_p	MA price mark-up	0.77	[0.64,0.92]	0.63	[0.44,0.83]
μ_w	MA wage mark-up	0.38	[0.21,0.57]	0.56	[0.32,0.80]
ρ_{ga}	$Cov(\sigma_a, \sigma_g)$	0.62	[0.46,0.76]	0.40	[0.22,0.57]

Table 5.10: Posterior estimates of parameters associated with the exogenous shocks for the pre- and post-Volcker period.

6 Conclusions

The paper studies the relevance of the interactions between monetary policy and the formation of expectations for U.S. business cycle fluctuations during the post-war period. I argue that a quantitative assessment of the mechanisms that rationalize the behavior of the data requires the adoption of a rich dynamic and stochastic structure such as the SW model. By implementing the methodology of Bianchi and Nicolò (2017), this paper constitutes, to the best of my knowledge, the first study that quantitatively investigates the role of self-fulfilling expectations and non-fundamental disturbances for the macroeconomic instability observed in the United States prior to 1979 in the context of a medium-scale model.

The data strongly supports the evidence of a passive monetary policy before 1979, even when accounting for richer propagation mechanisms and additional structural shocks. According to this monetary regime, the transmission of structural shocks is altered and crucially depends on the de-anchoring of expectations that instead are self-fulfilling.

The quantitative relevance of the role of self-fulfilling expectations and non-fundamental disturbances provides an explanation for U.S. business cycle that differs from previous studies such as SW in which these mechanisms are excluded *a priori*. While in the latter the run-up in inflation from the early 1960s to 1979 is attributed exclusively to mark-up shocks, the transmission mechanism based on self-fulfilling expectations provides an alternative explanation. Productivity shocks generated economic activity and self-fulfilling inflationary pressures that account for the rise in inflation in the 1960s. Mark-up shocks have quantitative importance to explain the sudden rise in inflation during the oil crisis of the 1970s. The high volatility before 1979 is explained by large structural shocks, while non-fundamental sunspot shocks play no quantitative role.

Extensions of this work would explore the possibility of accounting not only for dynamic indeterminacy, but also for static indeterminacy (i.e. multiplicity of steady states). Based on these cointegrating properties of the data (Beyer and Farmer, 2007b), Farmer and Platonov (2016) develop a micro-founded model that accounts for the possibility of observing multiple steady-state unemployment rates.[83] The three-equations version of the model corresponds

[83]Beyer and Farmer (2007b) study the low frequency comovements in unemployment, inflation and the federal funds rate and find that the data is well described by cointegrating relationships. This evidence raises doubts about the validity of the natural rate hypothesis on which conventional NK models such as SW rely

to the structural representation studied in Farmer (2012a). The model is equivalent to the conventional three-equation NK model in which the NK Phillips curve is replaced by a 'belief function' that describes how agents form expectations about future nominal income growth. In Farmer and Nicolò (2018), we show that the reduced-form representation corresponds to a cointegrated Vector Error Correction Model (VECM) and the model outperforms the conventional three-equation NK model in fitting the data before and after 1979.

An interesting avenue of research extends the proposed alternative framework to a medium-scale model that also displays multiplicity of steady states and therefore maps into a VECM in reduced-form. The purpose would therefore be to study whether the cointegrating properties of the proposed model would better explain the data in the post-war period relative to a conventional NK model that displays self-stabilizing properties around the unique steady state.

(i.e. the assumption that the long-run unemployment rate is independent of monetary and fiscal policies).

Appendices

A Appendix I.A

Proof. [Proof of Theorem 1] Let A^1 and A^2 be two orthonormal row operators associated with partitions $\mathbf{p}_1$ and $\mathbf{p}_2$;

$$\begin{bmatrix} z_t \\ \eta^1_{f,t} \\ \eta^1_{n,t} \end{bmatrix} = A^1 \begin{bmatrix} z_t \\ \eta_t \end{bmatrix}, \quad \begin{bmatrix} z_t \\ \eta^2_{f,t} \\ \eta^2_{n,t} \end{bmatrix} = A^2 \begin{bmatrix} z_t \\ \eta_t \end{bmatrix}. \tag{A1}$$

We assume that the operators, A^i have the form

$$A^i = \begin{bmatrix} \underset{l\times l}{I} & 0 \\ 0 & \underset{p\times p}{\tilde{A}^i} \end{bmatrix}, \tag{A2}$$

where $\tilde{A}^i$ is a permutation of the columns of an I_p identity matrix. Premultiplying the vector $[z_t, \eta_t]^T$ by the operator A^i permutes the rows of η_t while leaving the rows of z_t unchanged. Define matrices Ω_{ff} and Ω_{zf} for $i \in \{1, 2\}$ to be the new terms in the fundamental covariance matrix,

$$E\left(\begin{bmatrix} z_t \\ \eta^i_{f,t} \end{bmatrix} \begin{bmatrix} z_t \\ \eta^i_{f,t} \end{bmatrix}^T \right) = \begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix}.$$

Next, use (3.1) and (3.2) to write the non-fundamentals as linear functions of the fundamentals,

$$\eta^i_{n,t} = \Theta^i_z z_t + \Theta^i_f \eta^i_{f,t}, \tag{A3}$$

where

$$\Theta^i_z \equiv -\left(\tilde{\Pi}^i_{2n}\right)^{-1} \tilde{\Psi}_2, \quad \text{and} \quad \Theta^i_f \equiv -\left(\tilde{\Pi}^i_{2n}\right)^{-1} \tilde{\Pi}^i_{2f}, \tag{A4}$$

and define the matrix D^i,

$$D^i = \begin{bmatrix} \underset{l\times l}{I} & \underset{l\times m}{0} \\ \underset{m\times l}{0} & \underset{m\times m}{I} \\ \underset{(p-m)\times l}{\Theta_z^i} & \underset{(p-m)\times m}{\Theta_f^i} \end{bmatrix}. \tag{A5}$$

Using this definition, the covariance matrix of all shocks, fundamental and non-fundamental, has the following representation,

$$E\left(\begin{bmatrix} z_t \\ \eta_{f,t}^i \\ \eta_{n,t}^i \end{bmatrix}\begin{bmatrix} z_t \\ \eta_{f,t}^i \\ \eta_{n,t}^i \end{bmatrix}^T\right) = D^i \begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix} D^{iT}. \tag{A6}$$

We can also combine the last two row blocks of D^i and write D^i as follows

$$D^i = \begin{bmatrix} \underset{l\times l}{I} & \underset{l\times m}{0} \\ \underset{p\times l}{D_{21}^i} & \underset{p\times m}{D_{22}^i} \end{bmatrix}, \tag{A7}$$

where,

$$D_{21}^i = \begin{bmatrix} \underset{m\times l}{0} \\ \underset{(p-m)\times l}{\Theta_z^i} \end{bmatrix}, \quad D_{22}^i = \begin{bmatrix} \underset{m\times m}{I} \\ \underset{(p-m)\times m}{\Theta_f^i} \end{bmatrix}. \tag{A8}$$

Using (A1) and the fact that A^i is orthonormal, we can write the following expression for the complete set of shocks

$$\begin{bmatrix} z_t \\ \eta_t \end{bmatrix} = A^{iT} \begin{bmatrix} z_t \\ \eta_{f,t}^i \\ \eta_{n,t}^i \end{bmatrix}. \tag{A9}$$

Using equations (A6) and (A9), it follows that

$$E\left(\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}^T\right) = B^i W^i B^{iT}, \quad \text{for all} \quad \mathbf{p}_i \in \mathbf{P}, \tag{A10}$$

where

$$W^i \equiv \begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix}, \tag{A11}$$

and

$$B^i \equiv A^{iT} D^i = \begin{bmatrix} I & 0 \\ 0 & \tilde{A}^i \end{bmatrix}\begin{bmatrix} I & 0 \\ D^i_{21} & D^i_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^i_{21} & B^i_{22} \end{bmatrix}. \tag{A12}$$

Using this expression, we can write out equation (A10) in full to give,

$$E\left(\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}^T\right) = \begin{bmatrix} I & 0 \\ B^i_{21} & B^i_{22} \end{bmatrix}\begin{bmatrix} \Omega_{zz} & \Omega_{zf} \\ \Omega_{fz} & \Omega_{ff} \end{bmatrix}\begin{bmatrix} I & B^{iT}_{21} \\ 0 & B^{iT}_{22} \end{bmatrix}. \tag{A13}$$

We seek to establish that for any partition $\mathbf{p}_i$, parameterized by matrices Ω_{ff}, and Ω_{zf} that there exist matrices Ω_{ff} and Ω_{zf} for all partitions $\mathbf{p}_j \in \mathbf{P}, \quad j \neq i,$ such that

$$\Omega = E\left(\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}\begin{bmatrix} z_t \\ \eta_t \end{bmatrix}^T\right) = B^i W^i B^{iT} = B^j W^j B^{jT}. \tag{A14}$$

To establish this proposition, we write out the elements of (A13) explicitly. Since W^i and B^i are symmetric we need consider only the upper-triangular elements which give three equations in the matrices of Ω_{zf} and Ω_{ff},

$$\begin{aligned} \Omega_{11} &= \Omega_{zz}, \\ \Omega_{12} &= \Omega^i_{zz} B^{iT}_{21} + \Omega_{zf} B^{iT}_{22}, \\ \Omega_{22} &= B^i_{21}\Omega^i_{zz}B^{iT}_{21} + 2B^i_{21}\Omega_{zf}B^{iT}_{22} + B^i_{22}\Omega_{ff}B^{iT}_{22}. \end{aligned} \tag{A.1}$$

The first of these equations defines the covariance of the fundamental shocks and it holds for all i, j. Now define

$$a = vec\left(\Omega_{zz}\right), \quad x^i = vec\left(\Omega_{zf}\right), \quad y^i = vec\left(\Omega_{ff}\right). \tag{A16}$$

Using the fact that

$$vec\left(ABC\right) = \left(C^T \otimes A\right) \; vec\left(B\right), \tag{A17}$$

we can pass the *vec* operator through equation (A.1) and write the following system of linear equations in the unknowns x^j and y^j,

$$S^i \begin{bmatrix} x^i \\ y^i \end{bmatrix} + T^i a = S^j \begin{bmatrix} x^j \\ y^j \end{bmatrix} + T^j a, \tag{A18}$$

$$S^k = \begin{bmatrix} \left(B_{22}^k \otimes I\right) & 0 \\ \left(B_{22}^k \otimes B_{21}^j\right) & \left(B_{22}^k \otimes B_{22}^k\right) \end{bmatrix}, \quad T^k = \begin{bmatrix} \left(B_{21}^k \otimes I\right) \\ \left(B_{21}^k \otimes B_{21}^k\right) \end{bmatrix}, \quad k \in \{i, j\}. \tag{A19}$$

It follows from the assumption that the equilibrium is regular that S^j has full rank for all j hence for any permutation $\mathbf{p}_i$, parameterized by $\{x^i, y^i\}$we can find an alternative permutation $\mathbf{p}_j$ with associated parameterization $\{x^j, y^j\}$,

$$\begin{bmatrix} x^j \\ y^j \end{bmatrix} = \left(S^j\right)^{-1} \left(S^i \begin{bmatrix} x^i \\ y^i \end{bmatrix} + \left[T^i - T^j\right] a \right), \tag{A20}$$

that gives the same covariance matrix $\tilde{\Omega}$ for the fundamental and non-fundamental shocks.

□

B Appendix I.B

Proof. [Proof of Lemma 1] We seek to characterize the full set of solutions to the equation,

$$\underset{n\times\ell}{\tilde{\Psi}_2}\underset{\ell\times 1}{z_t} + \underset{n\times p}{\tilde{\Pi}_2}\underset{p\times 1}{\eta_t} = 0. \tag{B1}$$

Let U_1, V and D_{11} characterize the singular value decomposition of $\tilde{\Pi}_2$,

$$\underset{n\times p}{\tilde{\Pi}_2} \equiv \underset{n\times n}{U_1}\left[\begin{array}{cc} \underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}} \end{array}\right]\underset{p\times p}{V^T}, \tag{B2}$$

where we partition the matrix V as

$$V = \left[\begin{array}{cc} \underset{p\times n}{V_1} & \underset{p\times m}{V_2} \end{array}\right],$$

Let θ_{FKN} characterize a regular indeterminate equilibrium for some partition $\mathbf{p}_i$ and we partition η_t into two mutually exclusive subsets, $\eta^i_{f,t}$ and $\eta^i_{n,t}$ such that $\eta^i_{f,t} \cup \eta^i_{n,t} = \eta_t$. From Appendix A, equation A3, we write the non-fundamentals $\eta^i_{n,t}$ as functions of the fundamentals and where Θ^i_z and Θ^i_f are functions of θ_1,

$$\underset{n\times 1}{\eta^i_{n,t}} = \underset{n\times\ell}{\Theta^i_z}\underset{\ell\times 1}{z_t} + \underset{n\times m}{\Theta^i_f}\underset{m\times 1}{\eta^i_{f,t}}. \tag{B3}$$

Equation (B3) connects the non-fundamental shocks $\eta^i_{n,t}$ to the fundamental shocks $\left[z_t, \eta^i_{f,t}\right]$ in the FKN equilibrium. Equation (4.9) reproduced below as (B4), characterizes the additional equations that define an LS equilibrium,

$$\underset{p\times 1}{\eta_t} = \underset{p\times n}{V_1}\underset{n\times\ell}{N}\underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2}\underset{m\times\ell}{M_z}\underset{\ell\times 1}{z_t} + \underset{p\times m}{V_2}\underset{m\times 1}{\zeta_t}, \tag{B4}$$

where $N \equiv -D_{11}^{-1}U_1^T\tilde{\Psi}_2$. To establish the connection between the LS and FKN representations we split the equations of (B4) into two blocks

$$\underset{n\times 1}{\eta^i_{n,t}} = \underset{n\times n}{V^i_{1,n}} \underset{n\times \ell}{N} \underset{\ell\times 1}{z_t} + \underset{n\times m}{V^i_{2,n}} \underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{n\times m}{V^i_{2,n}} \underset{m\times 1}{\zeta_t} \tag{B5}$$

$$\underset{m\times 1}{\eta^i_{f,t}} = \underset{m\times n}{V^i_{1,f}} \underset{n\times \ell}{N} \underset{\ell\times 1}{z_t} + \underset{m\times m}{V^i_{2,f}} \underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{m\times m}{V^i_{2,f}} \underset{m\times 1}{\zeta_t} \tag{B6}$$

where for $j = 1, 2$, the matrices $V^i_{j,f}$ and $V^i_{j,n}$ are composed of the row vectors of V_j which, according to partition $\mathbf{p}_i$, correspond to the non-fundamental shocks included as fundamental, $\eta^i_{f,t}$, and those that are still non-fundamental, $\eta^i_{n,t}$.

Using (B3) to replacing $\eta^i_{n,t}$ in (B5) and combining with (B6)

$$\begin{bmatrix} \underset{n\times m}{\Theta^i_f} \\ I_m \end{bmatrix} \underset{m\times 1}{\eta^i_{f,t}} = \underset{p\times n}{V^i_1} \underset{n\times \ell}{N} \underset{\ell\times 1}{z_t} - \begin{bmatrix} \underset{n\times \ell}{\Theta^i_z} \\ \underset{m\times \ell}{\mathbf{0}} \end{bmatrix} \underset{\ell\times 1}{z_t} + \underset{p\times m}{V^i_2} \underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{p\times m}{V^i_2} \underset{m\times 1}{\zeta_t}, \tag{B7}$$

where

$$\underset{p\times n}{V^i_j} \equiv \begin{bmatrix} \underset{n\times n}{V^i_{j,n}} \\ \underset{m\times n}{V^i_{j,f}} \end{bmatrix}.$$

Premultiplying (B7) by $(V^i_2)^T$ and exploiting the fact that V is orthonormal, leads to the equation

$$\underset{m\times m}{G^i} \underset{m\times 1}{\eta^i_{f,t}} = \underset{m\times \ell}{H^i} \underset{\ell\times 1}{z_t} + \underset{m\times \ell}{M_z} \underset{\ell\times 1}{z_t} + \underset{m\times 1}{\zeta_t}, \tag{B8}$$

where

$$\underset{m\times m}{G^i} \equiv \underset{m\times p}{\left(V^i_2\right)^T} \underset{p\times m}{\begin{bmatrix} \underset{n\times m}{\Theta^i_f} \\ I_m \end{bmatrix}}, \quad \text{and} \quad \underset{m\times \ell}{H^i} \equiv \underset{m\times p}{\left(V^i_2\right)^T} \underset{p\times n}{V^i_1} \underset{n\times \ell}{N} - \underset{m\times p}{\left(V^i_2\right)^T} \underset{p\times l}{\begin{bmatrix} \underset{n\times \ell}{\Theta^i_z} \\ \underset{m\times \ell}{\mathbf{0}} \end{bmatrix}}. \tag{B9}$$

Rearranging (B8) and defining

$$\underset{m\times \ell}{S^i} \equiv \underset{m\times \ell}{H^i} + \underset{m\times \ell}{M_z} \tag{B10}$$

gives

$$\underset{m\times 1}{\zeta_t} = \underset{m\times m}{G^i}\,\underset{m\times 1}{\eta^i_{f,t}} - \underset{m\times\ell}{S^i}\,\underset{\ell\times 1}{z_t}\,, \tag{B11}$$

which is the expression we seek. □

C Appendix I.C

Proof. [Proof of Theorem 2] Let $\theta_{FKN} = \{\theta_1, \theta_2\}$ characterize an FKN equilibrium. From (B8), which we repeat below omitting the superscript i to reduce notation,

$$\underset{m\times m}{G}\,\underset{m\times 1}{\eta_{f,t}} = \underset{m\times\ell}{H}\,\underset{\ell\times 1}{z_t} + \underset{m\times\ell}{M_z}\,\underset{\ell\times 1}{z_t} + \underset{m\times 1}{\zeta_t}\,. \tag{C1}$$

Post-multiplying this equation by z_t^T and taking expectations gives

$$\underset{m\times m}{G}\,\underset{m\times\ell}{\Omega_{fz}} = \underset{m\times\ell}{H}\,\underset{\ell\times\ell}{\Omega_{zz}} + \underset{m\times\ell}{M_z}\,\underset{\ell\times\ell}{\Omega_{zz}} = \underset{m\times\ell}{S}\,\underset{\ell\times\ell}{\Omega_{zz}}, \tag{C2}$$

which represents $m\times\ell$ linear equations in the $m\times\ell$ elements of $vec\,(M_z)$ as functions of the elements of H, G and Ω_{zz}, (these are functions of θ_1), and Ω_{fz} (these are elements of θ_2). Applying the vec operator to (C2), using the algebra of Kronecker products, and rearranging terms gives the following solution for the parameters $vec\,(M_z)$,

$$\underset{(m\times\ell)\times 1}{vec\,(M_z)} =$$

$$\underset{(m\times\ell)\times(\ell\times m)}{(\Omega_{zz}\otimes I_m)^{-1}}\left[\underset{(m\times\ell)\times(\ell\times m)}{(I_\ell\otimes G)}\;\underset{(m\times\ell)\times 1}{vec\,(\Omega_{fz})} - \underset{(m\times\ell)\times\ell^2}{(I_\ell\otimes H)}\underset{\ell^2\times 1}{vec\,(\Omega_{zz})}\right]. \tag{C3}$$

Using equation (C3) we can construct an expression for the elements of S as functions of θ_1 and θ_2. Post-multiplying equation (B11) by itself transposed, and taking expectations, we have

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \underset{m\times m}{G}\,\underset{m\times m}{\Omega_{ff}}\,\underset{m\times m}{G^T} - \underset{m\times m}{G}\,\underset{m\times\ell}{\Omega_{fz}}\,\underset{\ell\times m}{S^T} - \underset{m\times\ell}{S}\,\underset{\ell\times m}{\Omega_{zf}}\,\underset{m\times m}{G^T} + \underset{m\times\ell}{S}\,\underset{\ell\times\ell}{\Omega_{zz}}\,\underset{\ell\times m}{S^T} \tag{C4}$$

$$= \underset{m\times m}{G}\underset{m\times m}{\Omega_{ff}}\underset{m\times m}{G^T} - \underset{m\times \ell}{S}\underset{\ell\times \ell}{\Omega_{zz}}\underset{\ell\times m}{S^T}$$

where the last equality is obtained using (C2). The terms on the RHS of (C4) are all functions of the known elements of θ_1 and θ_2. Since the matrix $\Omega_{\zeta\zeta}$ is symmetric, this gives $m \times (m+1)/2$ equations that determine the parameters of $vec(\Omega_{\zeta\zeta})$. This establishes that every $\theta_{FKN} \in \Theta_{FKN}$ defines a unique parameter vector $\theta_{LS} \in \Theta_{LS}$. To prove the converse, solve equation (C3) for $vec(\Omega_{fz})$ as a function of θ_1 and the elements of M_z and apply the vec operator to (C4) to solve for $vec(\Omega_{ff})$ in terms of θ_1 and $vec(\Omega_{\zeta\zeta})$. □

D Appendix I.D

To run the simulation of the New-Keynesian model in Lubik and Schorfheide (2004) under indeterminacy, we need to compute the matrices G^i, H^i and S^i. We proceed as follows. First, we apply the QZ decomposition to the representation of the model

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\mathbf{z}_t + \Pi(\theta)\eta_t, \tag{D1}$$

where $\Gamma_0(\theta)$, $\Gamma_1(\theta)$,$\Psi(\theta)$ and $\Pi(\theta)$ are described in Section 2. Let

$$\Gamma_0 = QSZ^T, \quad \text{and} \quad \Gamma_1 = QTZ^T, \tag{D2}$$

be the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$ where Q and Z are $k \times k$ orthonormal matrices and S and T are upper triangular and possibly complex. The resulting transformed parameters are

$$\tilde{\Psi} = Q^T\Psi, \quad \text{and} \quad \tilde{\Pi} = Q^T\Pi, \tag{D3}$$

which then allow us to define the equation connecting fundamental and non-fundamental errors

$$\underset{n\times\ell}{\tilde{\Psi}_2}\underset{\ell\times 1}{z_t} + \underset{n\times p}{\tilde{\Pi}_2}\underset{p\times 1}{\eta_t} = 0, \tag{D4}$$

where $\tilde{\Psi}_2$ and $\tilde{\Pi}_2$ are described in Section 2. For the New-Keynesian model in Lubik and Schorfheide (2004) the degree of indeterminacy $m = (p - n)$ equals 1 since the number of non-fundamental shocks is $p = 2$, while the number of generalized eigenvalues that are greater than or equal to 1 is $n = 1$.

Second, we follow Lubik and Schorfheide (2004) and apply the singular value decomposition as described in Section 4

$$\underset{n\times p}{\tilde{\Pi}_2} \equiv \underset{n\times n}{U_1}\left[\begin{array}{cc}\underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}}\end{array}\right]\underset{p\times p}{V^T}. \tag{D5}$$

and we compute

$$\underset{n\times\ell}{N} \equiv -\underset{n\times n}{D_{11}^{-1}}\underset{n\times n}{U_1^T}\underset{n\times\ell}{\tilde{\Psi}_2}. \tag{D6}$$

Third, we partition η_t into two mutually exclusive subsets, $\eta_{f,t}$ and $\eta_{n,t}$ such that $\eta_{f,t} \cup \eta_{n,t} = \eta_t$ and partition $\tilde{\Pi}_2$ conformably so that

$$\underset{n\times p}{\tilde{\Pi}_2}\underset{p\times 1}{\eta_t} = \left[\underset{n\times m}{\tilde{\Pi}^i_{2f}}\ \underset{n\times n}{\tilde{\Pi}^i_{2n}}\right]\left[\begin{array}{c}\underset{m\times 1}{\eta^i_{f,t}} \\ \underset{n\times 1}{\eta^i_{n,t}}\end{array}\right]. \tag{D7}$$

For the New-Keynesian model we are considering there are two possible partitions $i = \{1, 2\}$ for which we include the non-fundamental shock $\eta_{1,t} = x_t - E_{t-1}[x_t]$ or $\eta_{2,t} = \pi_t - E_{t-1}[\pi_t]$ respectively as fundamental shock. We then compute the matrices Θ^i_z and Θ^i_f as defined in

(A4) and which we report here

$$\Theta_z^i \equiv -\left(\tilde{\Pi}_{2n}^i\right)^{-1}\tilde{\Psi}_2, \quad \text{and} \quad \Theta_f^i \equiv -\left(\tilde{\Pi}_{2n}^i\right)^{-1}\tilde{\Pi}_{2f}^i \tag{D8}$$

Fourth, we partition V

$$V = \begin{bmatrix} \underset{p\times n}{V_1} & \underset{p\times m}{V_2} \end{bmatrix}, \tag{D9}$$

and define the matrices

$$\underset{p\times n}{V_j^i} \equiv \begin{bmatrix} \underset{n\times n}{V_{j,n}^i} \\ \underset{m\times n}{V_{j,f}^i} \end{bmatrix}, \tag{D10}$$

where the matrices $V_{j,f}^i$ and $V_{j,n}^i$ are composed of the row vectors of V_j which, according to partition $\mathbf{p}_i$, correspond to the non-fundamental shocks included as fundamental, $\eta_{f,t}^i$, and those that are still non-fundamental, $\eta_{n,t}^i$.

Finally, we use the definitions of G^i and H^i

$$\underset{m\times m}{G^i} \equiv \underset{m\times p}{\left(V_2^i\right)^T} \underset{p\times m}{\begin{bmatrix} \underset{n\times m}{\Theta_f^i} \\ I_m \end{bmatrix}}, \quad \text{and} \quad \underset{m\times \ell}{H^i} \equiv \underset{m\times p}{\left(V_2^i\right)^T} \underset{p\times n}{V_1^i} \underset{n\times \ell}{N} - \underset{m\times p}{\left(V_2^i\right)^T} \underset{p\times l}{\begin{bmatrix} \underset{n\times \ell}{\Theta_z^i} \\ \underset{m\times \ell}{\mathbf{0}} \end{bmatrix}}. \tag{D11}$$

for each partition $i = \{1, 2\}$. Therefore, we obtain the matrix

$$\underset{m\times \ell}{S^i} = \underset{m\times \ell}{H^i} + \underset{m\times \ell}{M_z}, \tag{D12}$$

where the $m \times \ell$ matrix M_z captures the correlation of the forecast errors with the fundamentals in Lubik and Schorfheide (2004) as explained in Section 4.1.

E Appendix II.A

In this Appendix, we show how the normalization chosen in Lubik and Schorfheide (2004) maps into the methodology we propose. Recall the following notation: p denotes number of expectational variables, n is the number of explosive roots and $m = (p - n)$ are the corresponding degrees of indeterminacy.

As in Lubik and Schorfheide (2004), consider the following structural model

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t \tag{E.1}$$

where X_t is the vector of endogenous variables, ε_t is the vector of exogenous shocks, η_t is the vector of endogenous shocks and we assume that the matrix Γ_0 is invertible. The system can therefore be written as

$$X_t = \Gamma_1^*(\theta)X_{t-1} + \Psi^*(\theta)\varepsilon_t + \Pi^*(\theta)\eta_t \tag{E.2}$$

Lubik and Schorfheide (2004) find that, if it exists, the solution to express the forecast errors as a function of the exogenous shocks ε_t and sunspot shocks ζ_t takes the form

$$\underset{p\times 1}{\eta_t} = \left(-\underset{p\times n}{V_1} \underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1^T} \underset{n\times \ell}{\tilde{\Psi}_2} + \underset{p\times m}{V_2} \underset{m\times \ell}{\tilde{M}} \right) \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times m}{M_\zeta} \underset{m\times 1}{\zeta_t}.$$

More compactly,

$$\underset{p\times 1}{\eta_t} = \underset{p\times n}{V_1} \underset{n\times \ell}{N} \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times \ell}{\tilde{M}} \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times m}{M_\zeta} \underset{m\times 1}{\zeta_t}, \tag{E.3}$$

where

$$\underset{n\times \ell}{N(\theta)} \equiv -\underset{n\times n}{D_{11}^{-1}(\theta)} \underset{n\times n}{U_1^T(\theta)} \underset{n\times \ell}{\tilde{\Psi}_2(\theta)}.$$

is a function of the parameters of the model.

Combining (E.2) with (E.3), the solution that appears in eq. (26) of their paper is

$$X_t = \Gamma_1^*(\theta)X_{t-1} + \left[\Psi^*(\theta) + \Pi^*(\theta)V_1(\theta)N(\theta)\right]\varepsilon_t \ + \Pi^*(\theta)V_2(\theta)\left(\tilde{M}\varepsilon_t + M_\zeta\zeta_t\right) \tag{E.4}$$

Determinacy Under determinacy, Lubik and Schorfheide (2004) show that $V_2(\theta) = 0$. Hence, the endogenous variables only respond to exogenous shocks. From (E.4) the solution is

$$X_t = \Gamma_1^*(\theta)X_{t-1} + \left[\Psi^*(\theta) + \Pi^*(\theta)V_1(\theta)N(\theta)\right]\varepsilon_t \tag{E.5}$$

Using the augmented representation that we propose in this paper, the solution under determinacy is equivalent to (E.5). Indeed, we are appending a stationary process which constitutes a separate block and does not interact with the endogenous variables of the model.

Indeterminacy Under indeterminacy, Lubik and Schorfheide (2004) show that $V_2(\theta) \neq 0$ and the endogenous variables not only respond to exogenous shocks but also to the sunspot shock ζ_t. Their solution is in eq. (E.4) and reported below

$$X_t = \Gamma_1^*(\theta)X_{t-1} + \left[\Psi^*(\theta) + \Pi^*(\theta)V_1(\theta)N(\theta)\right]\varepsilon_t \ + \Pi^*(\theta)V_2(\theta)\left(\tilde{M}\varepsilon_t + M_\zeta\zeta_t\right). \tag{E.6}$$

Now we consider the solution under indeterminacy that we obtain using our methodology. Also in our case we assume that Γ_0 is invertible and the system in (E.1) can be written as

$$X_t = \Gamma_1^*(\theta)X_{t-1} + \Psi^*(\theta)\varepsilon_t + \Pi^*(\theta)\eta_t. \tag{E.7}$$

Nevertheless, we can show that, if it exists, the solution to express to the endogenous shocks as a function of the exogenous shocks ε_t and sunspot shocks ν_t takes the form

$$\underset{p\times 1}{\eta_t} = \underset{p\times l}{C_1(\theta)}\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{C_2(\theta)}\underset{m\times 1}{\nu_t}. \tag{E.8}$$

where $\underset{p\times l}{C_1(\theta)} \equiv - \begin{bmatrix} \underset{n\times n}{\left(\widetilde{\Pi}_{n,2}(\theta)\right)^{-1}} \underset{n\times l}{\widetilde{\Psi}_2(\theta)} \\ \underset{m\times l}{\mathbf{0}} \end{bmatrix}$ and $\underset{p\times m}{C_1(\theta)} \equiv - \begin{bmatrix} \underset{n\times n}{\left(\widetilde{\Pi}_{n,2}(\theta)\right)^{-1}} \underset{n\times m}{\widetilde{\Pi}_{f,2}(\theta)} \\ \underset{m\times m}{-\mathbf{I}} \end{bmatrix}$.

Combining (E.7) and (E.8), we obtain the following reduced form

$$X_t = \Gamma_1^*(\theta)X_{t-1} + [\Psi^*(\theta) + \Pi^*(\theta)C_1(\theta)]\,\varepsilon_t \; + \Pi^*(\theta)C_2(\theta)\nu_t. \tag{E.9}$$

Identification As shown in the previous section, the solution under indeterminacy provided by Lubik and Schorfheide (2004) is derived by combining the following system of equations

$$X_t = \Gamma_1^*(\theta)X_{t-1} + \Psi^*(\theta)\varepsilon_t + \Pi^*(\theta)\eta_t \tag{E.10}$$

with the solution for the endogenous shocks as a function of the exogenous shocks ε_t and the sunspot shock η_t

$$\underset{p\times 1}{\eta_t} = \underset{p\times n}{V_1(\theta)}\underset{n\times \ell}{N(\theta)}\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2(\theta)}\underset{m\times \ell}{\tilde{M}}\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2(\theta)}\underset{m\times m}{M_\zeta}\underset{m\times 1}{\zeta_t}, \tag{E.11}$$

Similarly, the solution obtained using our methodology is derived by combining the same system of equations in (E.10) with our solution for the endogenous shocks

$$\underset{p\times 1}{\eta_t} = \underset{p\times l}{C_1(\theta)}\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{C_2(\theta)}\underset{m\times 1}{\nu_t}. \tag{E.12}$$

So, in order to understand how the identification strategy implemented in Lubik and Schorfheide (2004) maps into our solution, we only have to study the solutions for the endogenous shocks expressed in (E.11) and (E.12).

Lubik and Schorfheide (2004) consider a three equation NK model for which the degree of indeterminacy is at most 1 (i.e. $m = 1$). This implies that M_ζ, ζ_t and ν_t are scalars. Moreover, the authors assume the following two normalizations

$$\begin{aligned} E(\varepsilon_t \zeta_t') &= 0, &\text{(E.13)} \\ M_\zeta &= 1. &\text{(E.14)} \end{aligned}$$

To understand the mapping of these normalizations, we equate the RHS of (E.11) and (E.12),

$$\left(V_1 N \ + V_2 \tilde{M} \right) \varepsilon_t + V_2 M_\zeta \zeta_t \ = \ C_1 \varepsilon_t + C_2 \nu_t. \tag{E.15}$$

Pre-multiplying by V_2' and recalling that the matrix V_2 is orthonormal,

$$M_\zeta \zeta_t \ = \ \left(V_2' C_1 - V_2' V_1 N \ - \tilde{M} \right) \varepsilon_t + V_2' C_2 \nu_t. \tag{E.16}$$

Post-multiplying by ζ_t', we take expectation and consider the normalization in (E.13) to obtain

$$M_\zeta \sigma_\zeta^2 \ = \ (V_2' C_2)\, \sigma_{\nu\zeta}. \tag{E.17}$$

Noting that $\left(\underset{1\times p}{V_2'} \ \underset{p\times 1}{C_2} \right)$ is also a scalar

$$\sigma_{\nu\zeta} \ = \ \frac{M_\zeta}{(V_2' C_2)} \sigma_\zeta^2. \tag{E.18}$$

So, the normalization in (E.13) corresponds to specify a relationship between the covariance of the sunspot shock introduced in Lubik and Schorfheide (2004), ζ_t, and the sunspot shock that we specify in our methodology, ν_t, with the standard deviation of the sunspot shock, ζ_t, scaled by M_ζ. Therefore, the normalization $M_\zeta = 1$ in (E.14) is such that the relationship in (E.18) becomes

$$\sigma_{\nu\zeta} \;=\; \left(V_2' C_2\right)^{-1} \sigma_\zeta^2. \tag{E.19}$$

F Appendix II.B

We prove the equivalence between the parametrization of the Lubik-Schorfheide indeterminate equilibrium $\theta^{LS} \in \Theta^{LS}$ and the Bianchi-Nicolò equilibrium parametrized by $\theta^{BN} \in \Theta^{BN}$. In particular, we show that there is a unique mapping between the linear restrictions imposed in each of the two methodologies on the forecast errors to guarantee the existence of at least a bounded solution. As shown in Section 2.2.1, the method by Lubik and Schorfheide (2003) imposes the following restrictions on the non-fundamental shocks, η_t, as a function of the exogenous shocks, ε_t, and the sunspot shocks introduced in their specification, ζ_t,

$$\underset{p\times 1}{\eta_t} \;=\; \left(\underset{p\times n}{V_1}\,\underset{n\times \ell}{N} \;+\; \underset{p\times m}{V_2}\,\underset{\substack{m\times \ell\\ m\times \ell}}{\widetilde{M}}\right) \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2}\,\underset{m\times 1}{\zeta_t}\,. \tag{F.1}$$

Using the methodology proposed in this paper, Section 2.2.2 shows that the restrictions on the non-fundamental shocks, η_t, as a function of the exogenous shocks, ε_t, and the sunspot shocks, υ_t, are

$$\underset{p\times 1}{\eta_t} = \underset{p\times \ell}{C_1}\,\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{C_2}\,\underset{m\times 1}{\nu_t}\,, \tag{F.2}$$

where

$$C_1 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2 \\ \mathbf{0} \end{bmatrix} \qquad \text{and} \qquad C_2 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2} \\ -\mathbf{I} \end{bmatrix}.$$

Post-multiplying equation (F.1) and (F.2) by ε_t^T and taking expectation,

$$\begin{aligned}
\underset{p\times l}{\Omega_{\eta\varepsilon}} &= \underset{p\times n}{V_1}\,\underset{n\times \ell}{N}\,\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{p\times m}{V_2}\,\underset{m\times \ell}{\widetilde{M}}\,\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}},\\
\underset{p\times l}{\Omega_{\eta\varepsilon}} &= \underset{p\times \ell}{C_1}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{p\times m}{C_2}\,\underset{m\times l}{\Omega_{\nu\varepsilon}}
\end{aligned}$$

Pre-multiplying by V_2^T and equating the equations,

$$\underset{m\times\ell}{\widetilde{M}}\,\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} = \left(\underset{m\times p}{V_2^T}\,\underset{p\times\ell}{C_1} - \underset{m\times p}{V_2^T}\,\underset{p\times n}{V_1}\,\underset{n\times\ell}{N}\right)\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{m\times p}{V_2^T}\,\underset{p\times m}{C_2}\,\underset{m\times l}{\Omega_{\nu\varepsilon}}.$$

Using the properties of the vec operator, the following result holds

$$\underset{(m\times\ell)\times 1}{vec(\widetilde{M})} = \underset{(m\times\ell)\times(m\times\ell)}{(\Omega_{\varepsilon\varepsilon}\otimes I_m)^{-1}}\left[\underset{(m\times\ell)\times\ell^2}{\left[I_l\otimes\left(V_2^TC_1 - V_2^TV_1N\right)\right]}\underset{\ell^2\times 1}{vec\left(\Omega_{\varepsilon\varepsilon}\right)} + \underset{(m\times\ell)\times(m\times\ell)}{\left(I_l\otimes V_2^TC_2\right)}\underset{(m\times\ell)\times 1}{vec\left(\Omega_{\nu\varepsilon}\right)}\right]. \tag{F.3}$$

Also, considering again equation (F.1) and (F.2), we post-multiply by ζ_t^T and take expectation,

$$\begin{aligned}\underset{p\times m}{\Omega_{\eta\zeta}} &= \underset{p\times m}{V_2}\,\underset{m\times m}{\Omega_{\zeta\zeta}}\,,\\ \underset{p\times m}{\Omega_{\eta\zeta}} &= \underset{p\times m}{C_2}\,\underset{m\times m}{\Omega_{\nu\zeta}}\end{aligned}$$

Pre-multiplying both equations by V_2^T and equating them,

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \underset{m\times m}{\Omega_{\zeta\nu}}\,\underset{m\times m}{\left(V_2^TC_2\right)^T}. \tag{F.4}$$

Finally, to obtain an expression for $\Omega_{\zeta\nu}$, we post-multiply equation (F.1) and (F.2) by ν_t^T and taking expectations

$$\begin{aligned}\underset{p\times m}{\Omega_{\eta\nu}} &= \left(\underset{p\times n}{V_1}\,\underset{n\times\ell}{N} + \underset{p\times m}{V_2}\,\underset{m\times\ell}{\widetilde{M}}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{p\times m}{V_2}\,\underset{m\times m}{\Omega_{\zeta\nu}}\,,\\ \underset{p\times m}{\Omega_{\eta\nu}} &= \underset{p\times\ell}{C_1}\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{p\times m}{C_2}\,\underset{m\times m}{\Omega_{\nu\nu}}\end{aligned}$$

Pre-multiplying both equations by V_2^T and solving for $\Omega_{\zeta\nu}$,

$$\underset{m\times m}{\Omega_{\zeta\nu}} = \left(\underset{m\times p}{V_2^T}\,\underset{p\times\ell}{C_1} - \underset{m\times p}{V_2^T}\,\underset{p\times n}{V_1}\,\underset{n\times\ell}{N} - \underset{m\times\ell}{\widetilde{M}}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{m\times m}{\left(V_2^TC_2\right)}\,\underset{m\times m}{\Omega_{\nu\nu}}\,. \tag{F.5}$$

Post-multiplying (F.5) by $\underset{m\times m}{\left(V_2^T C_2\right)^T}$ and using (F.4), then

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \Big(\underset{m\times p}{V_2^T}\underset{p\times\ell}{C_1} - \underset{m\times p}{V_2^T}\underset{p\times n}{V_1}\underset{n\times\ell}{N} - \underset{m\times\ell}{\widetilde{M}}\Big)\underset{\ell\times m}{\Omega_{\varepsilon\nu}}\underset{m\times m}{\left(V_2^T C_2\right)^T} + \underset{m\times m}{\left(V_2^T C_2\right)}\underset{m\times m}{\Omega_{\nu\nu}}\underset{m\times m}{\left(V_2^T C_2\right)^T}. \tag{F.6}$$

Therefore, equation (F.3) and (F.6) define the one-to-one mapping between the parametrization in Lubik and Schorfheide $\{\Theta, \Theta^{LS}\}$ and the parametrization in Bianchi and Nicolò $\{\Theta, \Theta^{BN}\}$.

G Appendix II.C

In this section, we provide the derivation for the solutions under the two alternative representations discussed in Section 3.1 are provided.

a) Under determinacy, it is possible to use standard solution algorithms, such as Sims (2001b).

Consider the three equations NK model in (3.1)∼(3.3) and reported below as equations (G.1)∼(G.5)

$$x_t = E_t(x_{t+1}) - \tau(R_t - E_t(x_{t+1})) \tag{G.1}$$

$$\pi_t = \beta E_{t-1}(\pi_{t+1}) + \kappa x_t \tag{G.2}$$

$$R_t = \psi\pi_t + \varepsilon_{R,t} \tag{G.3}$$

$$\eta_{1,t} = x_t - E_{t-1}(x_t) \tag{G.4}$$

$$\eta_{2,t} = \pi_t - E_{t-1}(\pi_t) \tag{G.5}$$

The LRE model can be written in the following matrix form

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi z_t + \Pi\eta_t, \tag{G.6}$$

where $X_t = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}))^T$, $\varepsilon_t = (\varepsilon_{R,t})$ and $\eta_t = (\eta_{1,t}, \eta_{2,t})^T$.

The solution to (G.6) can be found following four steps. First, since the matrix Γ_0 is non-singular, the LRE model in (G.6) can be written as

$$X_t = \Gamma_1^* X_{t-1} + \Psi^* \varepsilon_t + \Pi^* \eta_t, \tag{G.7}$$

where

$$\Gamma_1^* \equiv \Gamma_0^{-1}\Gamma_1 = \begin{bmatrix} \mathbf{0_{2\times 2}} & \mathbf{I_{2\times 2}} \\ \mathbf{0_{2\times 2}} & A_{2\times 2} \end{bmatrix}, \qquad A \equiv \begin{bmatrix} 1+\frac{\kappa\tau}{\beta} & \tau\left(\psi - \frac{1}{\beta}\right) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

$$\Psi^* \equiv \Gamma_0^{-1}\Psi = \begin{bmatrix} 0 \\ 0 \\ \tau \\ 0 \end{bmatrix}, \qquad \Pi^* \equiv \Gamma_0^{-1}\Pi = \begin{bmatrix} \mathbf{I_{2\times 2}} \\ A_{2\times 2} \end{bmatrix}$$

Equivalently, the equations in (43) are

$$x_t = E_{t-1}(x_t) + \eta_{1,t} \tag{G.8}$$

$$\pi_t = E_{t-1}(\pi_t) + \eta_{2,t} \tag{G.9}$$

$$\xi_t = A\xi_{t-1} + \begin{bmatrix} \tau \\ 0 \end{bmatrix} \varepsilon_{R,t} + A\eta_t \tag{G.10}$$

where $\xi_t = (E_t(x_{t+1}), E_t(\pi_{t+1}))^T$.

Second, in order to study the stability of the system, the matrix A is decomposed using the Jordan decomposition[84] and (G.10) can be written as

[84] The Jordan decomposition of the matrix A is $A \equiv J\Lambda J^{-1}$, where the diagonal elements of the matrix Λ are the roots of the system.

$$J^{-1}\xi_t = \Lambda J^{-1}\xi_{t-1} + J^{-1}\begin{bmatrix}\tau \\ 0\end{bmatrix}\varepsilon_{R,t} + J^{-1}A\eta_t, \tag{G.11}$$

where

$$J^{-1} = \begin{bmatrix} -\frac{\kappa}{\phi} & -\frac{a_2}{2\phi} \\ \frac{\kappa}{\phi} & \frac{\beta+\phi+\kappa\tau-1}{2\phi} \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \lambda_{1,2} = \frac{(1+\beta+\kappa\tau)\pm\phi}{2\beta} \tag{G.12}$$

where $a_2 \equiv (\beta - \phi + \kappa\tau - 1)$, $\phi \equiv [(1+\beta+\kappa\tau)^2 - 4\beta(1+\kappa\tau\psi)]^{-1/2}$ and the diagonal elements of the matrix Λ are the roots of the system and under determinacy $|\lambda_{1,2}| > 1$.

Third, restrictions which eliminate the explosive dynamics of the system have to be imposed. Under determinacy both roots of (G.11) are unstable, which requires to impose the following conditions

$$\xi_t = \begin{pmatrix} E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix} = \mathbf{0_{2\times 1}} \tag{G.13}$$

$$\eta_t = -A^{-1}\begin{bmatrix}\tau \\ 0\end{bmatrix}\varepsilon_{R,t} = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1 \\ \kappa\end{bmatrix}\varepsilon_{R,t} \tag{G.14}$$

Fourth, the restrictions imposed on the endogenous variables and on the forecast errors are combined with the equations which define the remaining endogenous variables, that is (G.8) and (G.9). This implies

$$\begin{pmatrix} x_t \\ \pi_t \end{pmatrix} = \eta_t = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix}1 \\ \kappa\end{bmatrix}\varepsilon_{R,t}. \tag{G.15}$$

b) Here we provide the derivation for the solution in Section 3.1 for the methodology proposed in this paper.

The proposed methodology consists in appending to the original LRE model the following equation[85]

$$\omega_t = \frac{1}{\alpha}\omega_{t-1} + \nu_t - \eta_{2,t},$$

where without loss of generality $\alpha \equiv \psi > 1$. Denoting the newly defined vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)^T = (x_t, \pi_t, E_t(x_{t+1}), E_t(\pi_{t+1}), \omega_t)^T$, and the newly defined vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \upsilon_t)^T = (\varepsilon_{R,t}, \upsilon_t)^T$, the augmented representation of the LRE model is

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{z}_t + \hat{\Pi}\eta_t. \tag{G.16}$$

Given (G.16), the same steps are followed to obtain the solution to the system. First, the system in (G.16) is pre-multiplied by $\hat{\Gamma}_0^{-1}$ to obtain

$$\hat{X}_t = \hat{\Gamma}_1^* \hat{X}_{t-1} + \hat{\Psi}^*\hat{\varepsilon}_t + \hat{\Pi}^*\eta_t, \tag{G.17}$$

where

$$\hat{\Gamma}_1^* \equiv \begin{bmatrix} \Gamma_1^* & \mathbf{0_{4\times 1}} \\ & \\ \mathbf{0_{1\times 4}} & \frac{1}{\alpha} \end{bmatrix}, \quad \hat{\Psi}^* \equiv \begin{bmatrix} \Psi^* & \mathbf{0_{4\times 1}} \\ & \\ 0 & -1 \end{bmatrix}, \quad \hat{\Pi}^* \equiv \begin{bmatrix} & \Pi_{4\times 2}^* \\ & \\ 0 & 1 \end{bmatrix}.$$

[85]Note that $m = 1$, thus implying that only one equation should be appended. Also, since Farmer et al. (2015) show that the choice of which forecast errors should be redefined as fundamental, it is without loss of generality that we consider the case when $\eta_{2,t}$ is redefined.

Hence, defining $\hat{\xi}_t \equiv (\xi_t, \omega_t)^T = (E_t(x_{t+1}), E_t(\pi_{t+1}), \omega_t)^T$, the equations in (G.17) can be written as

$$x_t = E_{t-1}(x_t) + \eta_{1,t} \tag{G.18}$$

$$\pi_t = E_{t-1}(\pi_t) + \eta_{2,t} \tag{G.19}$$

$$\hat{\xi}_t = \hat{A} \quad \hat{\xi}_{t-1} + \begin{bmatrix} \tau & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \hat{z}_t + \begin{bmatrix} & A_{2\times 2} \\ & \\ 0 & -1 \end{bmatrix} \eta_t \tag{G.20}$$

where $\hat{A} = \begin{bmatrix} A & 0 \\ 0 & \alpha \end{bmatrix}$.

Second, the matrix $\hat{A}$ is decomposed using the Jordan decomposition and the system in (G.20) can be written as

$$\hat{J}^{-1}\hat{\xi}_t = \hat{\Lambda}\hat{J}^{-1}\hat{\xi}_{t-1} + \hat{J}^{-1} \begin{bmatrix} \tau & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \hat{z}_t + \hat{J}^{-1} \begin{bmatrix} & A_{2\times 2} \\ & \\ 0 & -1 \end{bmatrix} \eta_t, \tag{G.21}$$

where

$$\hat{J}^{-1} \equiv \begin{bmatrix} \mathbf{0_{1\times 2}} & 1 \\ & \\ J^{-1} & \mathbf{0_{2\times 1}} \end{bmatrix}, \quad \hat{\Lambda} \equiv \begin{bmatrix} \lambda_3 & 0 \\ 0 & \Lambda \end{bmatrix} = \begin{bmatrix} \lambda_3 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_2 \end{bmatrix}$$

and $\lambda_{1,2}$ are the same as in (G.12) and $\lambda_3 = (1/\alpha) = (1/\psi) < 1$.

Third, since $|\lambda_{1,2}| > 1$ and $\lambda_3 < 1$, then the conditions which guarantee the boundedness of the solution are imposed on the last two equations of (G.21). This implies

$$\xi_t = \begin{pmatrix} E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix} = \mathbf{0_{2\times 1}} \tag{G.22}$$

$$\eta_t = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix} 1 & 0 \\ \kappa & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_{R,t} \\ \nu_t \end{bmatrix} \tag{G.23}$$

Fourth, combining these restrictions with the first equation of (G.21) which displays stable dynamics and with (G.18) and (G.19), the obtained solution is

$$\omega_t = \frac{1}{\alpha}\omega_{t-1} + \begin{bmatrix} \frac{\tau\kappa}{1+\kappa\tau\psi} & 1 \end{bmatrix}\begin{bmatrix} \varepsilon_{R,t} \\ \nu_t \end{bmatrix} \tag{G.24}$$

$$\begin{pmatrix} x_t \\ \pi_t \end{pmatrix} = \eta_t = -\frac{\tau}{1+\kappa\tau\psi}\begin{bmatrix} 1 & 0 \\ \kappa & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_{R,t} \\ \nu_t \end{bmatrix}. \tag{G.25}$$

H Appendix II.D

In Section 3.2, the NK model is indeterminate and the derivations for the solutions under two alternative representations are provided.

c) To select a unique, bounded rational expectation equilibrium, we follow the solution method suggested by Farmer et al. (2015) when the forecast error for the deviations of inflation from its steady state is included as newly defined fundamental shock. Defining $\tilde{\varepsilon}_t = (\varepsilon_t, \eta_{2,t})^T$, then the LRE can be written as

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi_f \tilde{\varepsilon}_t + \Pi_n \eta_{1,t}. \tag{H.1}$$

The same steps as in Section F are also applied here. First, by pre-multiplying (H.1) by Γ_0^{-1}, we obtain the following equations

$$
\begin{aligned}
x_t &= E_{t-1}(x_t) + \eta_{1,t} && \text{(H.2)}\\
\pi_t &= E_{t-1}(\pi_t) + \eta_{2,t} && \text{(H.3)}\\
\xi_t &= A\xi_{t-1} + \begin{bmatrix} \tau & \tau\left(\psi - \frac{1}{\beta}\right) \\ 0 & \frac{1}{\beta} \end{bmatrix} \tilde{\varepsilon}_t + \begin{bmatrix} 1 + \frac{\kappa\tau}{\beta} \\ -\frac{\kappa}{\beta} \end{bmatrix} \eta_{1t} && \text{(H.4)}
\end{aligned}
$$

where the matrix A is the same as for the determinate case as defined in (G.10) and therefore also its Jordan decomposition delivers the same matrices J and Λ as in (G.11) and (G.12) and reported below

$$
\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \lambda_{1,2} = \frac{(1+\beta+\kappa\tau) \pm \phi}{2\beta} \tag{H.5}
$$

and

$$
J^{-1} = \begin{bmatrix} -\frac{\kappa}{\phi} & -\frac{a_2}{2\phi} \\ \frac{\kappa}{\phi} & \frac{\beta+\phi+\kappa\tau-1}{2\phi} \end{bmatrix}.
$$

However, the difference with the determinate case is that, while in the latter both roots are outside the unit circle, under indeterminacy it is the case that $|\lambda_1| > 1$ and $|\lambda_2| < 1$. This implies that in the third step the restrictions imposed on the system to guarantee a bounded solution are also distinct from the determinate case. In particular, the restrictions are imposed on the first equation of (H.4), thus obtaining the following conditions

$$
E_t(x_{t+1}) = -\frac{a_2}{2\kappa} E_t(\pi_{t+1}) \tag{H.6}
$$

$$
\eta_{1,t} = \begin{bmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \end{bmatrix} \begin{bmatrix} \varepsilon_{R,t} \\ \eta_{2,t} \end{bmatrix} \tag{H.7}
$$

where $a_1 = (\beta - \phi + \kappa\tau + 1)$, $a_2 = (a_1 - 2)$, $a_3 = (a_1 + 2\phi)$ and $\phi = [(1+\beta+\kappa\tau)^2 - 4\beta(1+\kappa\tau\psi)]^{-1/2}$.

Fourth, using these restrictions, the solution obtained with the methodology of Farmer et al. (2015) is

$$\begin{pmatrix} x_t \\ \pi_t \\ E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix} = \underset{4\times 1}{G} E_{t-1}(\pi_t) + \underset{4\times 2}{H} \begin{bmatrix} \varepsilon_{R,t} \\ \eta_{2,t} \end{bmatrix} \tag{H.8}$$

where

$$\underset{4\times 1}{G} \equiv \begin{pmatrix} -\frac{a_2}{2\kappa} \\ 1 \\ -\frac{a_1 a_2}{4\beta\kappa} \\ \frac{a_1}{2\beta} \end{pmatrix} \qquad \underset{4\times 2}{H} \equiv \begin{pmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \\ 0 & 1 \\ -\frac{\tau a_2}{a_3} & -\frac{a_2(1+\kappa\tau\psi)}{a_3\kappa} \\ \frac{2\kappa\tau}{a_3} & -\frac{2(1+\kappa\tau\psi)}{a_3} \end{pmatrix}.$$

d) The derivation of the solution provided by the proposed methodology when the model is indeterminate closely follows the one described in Appendix B, part b). In particular, the first two steps of the solution method are equivalent and, recalling the definition of $\hat{\xi}_t \equiv (\xi_t, \omega_t)^T = (E_t(x_{t+1}), E_t(\pi_{t+1}), \omega_t)^T$ and $\hat{\varepsilon}_t \equiv (\varepsilon_t, v_t)^T = (\varepsilon_{R,t}, v_t)^T$, equation (G.21) is reported below as (H.9)

$$\hat{J}^{-1}\hat{\xi}_t = \hat{\Lambda}\hat{J}^{-1}\hat{\xi}_{t-1} + \hat{J}^{-1} \begin{bmatrix} \tau & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \hat{\varepsilon}_t + \hat{J}^{-1} \begin{bmatrix} A_{2\times 2} & 0 \\ 0 & -1 \end{bmatrix} \eta_t, \tag{H.9}$$

where

$$\hat{J}^{-1} \equiv \begin{bmatrix} \mathbf{0_{1\times 2}} & 1 \\ J^{-1} & \mathbf{0_{2\times 1}} \end{bmatrix}, \quad \hat{\Lambda} = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \Lambda \end{bmatrix} = \begin{bmatrix} \lambda_3 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_2 \end{bmatrix}.$$

It is however important to note that under indeterminacy not only $|\lambda_1| > 1$ and $|\lambda_2| < 1$ as in representation c), but also $|\lambda_3| = (1/\alpha) = (1/\psi) > 1$. Hence, the third steps imposes restrictions on the first two equations of (H.9), which result in the following conditions

$$\omega_t = 0 \tag{H.10}$$

$$E_t(x_{t+1}) = -\frac{a_2}{2\kappa}E_t(\pi_{t+1}) \tag{H.11}$$

$$\eta_t = \begin{bmatrix} -\frac{2\beta\tau}{a_3} & \frac{2\kappa\tau(1-\beta\psi)-a_2}{a_3\kappa} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{R,t} \\ \nu_t \end{bmatrix} \tag{H.12}$$

Fourth, using these restrictions, the solution of the LRE model for the endogenous variables takes the following form

$$\begin{pmatrix} x_t \\ \pi_t \\ E_t(x_{t+1}) \\ E_t(\pi_{t+1}) \end{pmatrix} = \underset{4\times1}{G} E_{t-1}(\pi_t) + \underset{4\times2}{H} \begin{bmatrix} \varepsilon_{R,t} \\ \nu_t \end{bmatrix}. \tag{H.13}$$

I Appendix III.A

The Reduced Forms of the NK and FM Models

In Appendix I we find solutions to simplified versions of the two models and we show how they are different from each other. To find closed form solutions, we set $\rho = 0$, $\eta = 0$, $a = 1$, $\bar{r} = 0$ and $\rho_R = 0$. These simplifications allow us to solve the models by hand using a Jordan decomposition. For more general parameter values we rely on numerical solutions that we compute using Christopher Sim's code, GENSYS Sims (2001b).

Solving the NK Model

Consider the following stripped down version of the NK model

$$\begin{aligned}
y_t &= E_t[y_{t+1}] - (R_t - E_t[\pi_{t+1}]) \\
R_t &= \lambda \pi_t + \mu y_y + z_{R,t} \\
\pi_t &= \beta E_{t-1}(\pi_{t+1}) + \phi y_t \\
\nu_{1,t} &= y_t - E_{t-1}(y_t) \\
\nu_{2,t} &= \pi_t - E_{t-1}(\pi_t)
\end{aligned}$$

The model can be written in the following matrix form

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi \varepsilon_t + \Pi \nu_t, \tag{I.1}$$

where $X_t \equiv (y_t, \pi_t, E_t(y_{t+1}), E_t(\pi_{t+1}))^\top$, $\varepsilon_t = (z_{R,t})$ and $\nu_t = (\nu_{1,t}, \nu_{2,t})^\top$.

Defining the matrix $\Gamma_1^* \equiv \Gamma_0^{-1}\Gamma_1$ we may rewrite this equation,

$$X_t = \Gamma_1^* X_{t-1} + \Psi^* \varepsilon_t + \Pi^* \nu_t. \tag{I.2}$$

The existence of a unique bounded solution to Equation (I.2) requires that two roots of the matrix Γ_1^* are outside the unit circle. This condition is satisfied when the following generalized form of the Taylor Principle holds,

$$\left|\lambda + \frac{1-\beta}{\phi}\mu\right| > 1.$$

In this case, the reduced form is an equation,

$$X_t = G^{NK} X_{t-1} + H^{NK} z_{R,t} \tag{I.3}$$

where H^{NK} is a 5×1 vector of coefficients and G^{NK} is a 5×5 matrix of zeros.

When the Taylor Principle breaks down, one or more elements of the vector of non-fundamental shocks, ν_t, can be reclassified as fundamental. In the case considered in this paper and favored by the data, the reduced form can be represented as

$$X_t = G^{NK} X_{t-1} + H^{NK} \begin{bmatrix} z_{R,t} \\ \nu_{2,t} \end{bmatrix} \tag{I.4}$$

where H^{NK} is a 5×2 vector of coefficients and G^{NK} is a 5×5 matrix of rank 4.

Solving the FM model

The equivalent stripped-down version of the FM model is represented by the equations,

$$\begin{aligned} y_t &= \mathbb{E}_t[y_{t+1}] - \left(R_t - \mathbb{E}_t[\pi_{t+1}]\right), \\ R_t &= \lambda \pi_t + \mu y_t + z_{R,t}, \\ x_t &= \mathbb{E}_t[x_{t+1}], \\ x_t &= y_t - y_{t-1} + \pi_t, \\ \nu_{x,t} &= x_t - E_{t-1}(x_t). \end{aligned}$$

Using the definition of x_t and the Taylor Rule to eliminate π_t and R_t. this can be rewritten as a system of three equations in the variables x_t, y_t and $\mathbb{E}_t[x_{t+1}]$,

$$\begin{aligned}
\mathbb{E}_t[x_{t+1}] - \lambda x_t - (\mu - \lambda) y_t &= \lambda y_{t-1} + z_{R,t} \\
\mathbb{E}_t[x_{t+1}] - x_t &= 0, \\
x_t &= E_{t-1}(x_t) + \nu_{x,t}.
\end{aligned}$$

In matrix notation

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi \varepsilon_t + \Pi \nu_{x,t},$$

where $X_t \equiv (\mathbb{E}_t[x_{t+1}], x_t, y_t)$, $\varepsilon_t = (z_{R,t})^\top$ and

$$\Gamma_0 = \begin{bmatrix} 1 & \lambda & \lambda - \mu \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Pi = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{I.5}$$

For this example the matrix Γ_0 is non-singular and the system can be written as

$$X_t = \Gamma_1^* X_{t-1} + \Psi^* \varepsilon_t + \Pi^* \nu_{x,t}$$

where $\Gamma_1^* \equiv \Gamma_0^{-1} \Gamma_1$, $\Psi^* \equiv \Gamma_0^{-1} \Psi$ and $\Pi^* \equiv \Gamma_0^{-1} \Pi$.

The matrix Γ_1^* is given by the expression

$$\Gamma_1^* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -\frac{1+\lambda}{\lambda-\mu} & 0 & \frac{\lambda}{\lambda-\mu} \end{bmatrix},$$

which has eigenvalues $\{0, 1, \frac{\lambda}{\lambda-\mu}\}$. Since the model has one non-fundamental shock, $\nu_{x,t}$, the analog of the Taylor Principle requires one root of Γ_1^* to lie outside the unit circle, that is

$$\left|\frac{\lambda}{\lambda - \mu}\right| > 1.$$

When the Taylor Principle holds, the reduced form is an equation of the form

$$X_t = G^{FM} X_{t-1} + H^{FM} z_{R,t} \tag{I.6}$$

where G^{FM} is constructed by choosing $\nu_{x,t}$ to eliminate the unstable root. The resulting matrix, G^{FM} two zero roots and a root of unity.

J Appendix III.B

Dynamic Properties for generalized IS curve

We now show that the dynamic properties of the FM model depend not only on the parameters of the monetary policy reaction function but importantly also on the parameter of relative risk aversion a. To simplify the notation, we consider the case of $\rho_R = 0$ and proceed to solve the model as in Appendix A. The roots of the system are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$ and

$$\lambda_{4,5} = \frac{-(\lambda - \mu - a + 1) \pm \sqrt{(\lambda - \mu - a + 1)^2 + 4\lambda(a - 1)}}{2(a - 1)}.$$

Given the posterior mean of the parameter $\lambda = 0.92$ and $\mu = 0.99$, we focus on the approximated roots for $(\lambda - \mu) = 0$. Thus, we obtain

$$\lambda_{4,5} = \frac{(a - 1) \pm \sqrt{(-a + 1)^2 + 4\lambda(a - 1)}}{2(a - 1)} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4\lambda}{(a - 1)}}.$$

We first show that the eigenvalue $\lambda_4 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\lambda}{(a-1)}}$ is always unstable for realistic values of the parameter λ and a. If $(a - 1) > 0$, then $\lambda_4 > 1$. If $(a - 1) < 0$, then $0 < \lambda_4 < 1$

if and only if $4\lambda < (1-a)$ or equivalently $a < 1 - 4\lambda$. For realistic values of the parameter λ, this is never the case, implying that λ_4 is always an unstable root of the model.

Given that the FM model has two forward-looking variables and that $\lambda_4 > 1$, the model is dynamically determinate if $\lambda_5 = \left[\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4\lambda}{(a-1)}}\right] < -1$. Simplifying, this condition can be written as

$$a < 1 + \frac{\lambda}{2}.$$

The posterior means reported in Table 3 and 4 for both the pre- and post-Volcker period indicate that this condition is violated, and that the dynamic properties of the FM model crucially depend on the value of the parameter a.

K Appendix IV.A

Appendix A reports the prior distributions of the structural parameters and volatility of the shocks used for the estimation of the SW model.

Coefficient	Description	Distr.	Mean	Std. Dev
ϕ	Adjustment cost	Normal	4.00	1.50
σ_c	IES	Normal	1.50	0.37
h	Habit Persistence	Beta	0.70	0.10
σ_l	Labor supply elasticity	Normal	2.00	0.75
ξ_w	Wage stickiness	Beta	0.50	0.10
ξ_p	Price Stickiness	Beta	0.50	0.10
ι_w	Wage Indexation	Beta	0.50	0.15
ι_p	Price Indexation	Beta	0.50	0.15
ψ	Capacity utilization elasticity	Beta	0.50	0.15
Φ	Share of fixed costs	Normal	1.25	0.12
α	Share of capital	Normal	0.30	0.05
$\bar{\pi}$	S.S. inflation rate (quart.)	Gamma	0.62	0.10
$100(\beta^{-1} - 1)$	Discount factor	Gamma	0.25	0.10
$\bar{l}$	S.S. hours worked	Normal	0.00	2.00
$\bar{\gamma}$	Trend growth rate	Normal	0.40	0.10
r_π	Taylor rule inflation	Normal	***1.00***	***0.35***
r_y	Taylor rule output gap	Normal	0.12	0.05
$r_{\Delta y}$	Taylor rule Δ(output gap)	Normal	0.12	0.05
ρ	Taylor rule smoothing	Beta	0.75	0.10

Table K.1: Prior distributions for the structural parameters of the model.

Coefficient	Description	Distr.	Mean	Std. Dev
σ_a	Technology shock	Invgamma	0.10	2.00
σ_b	Risk premium shock	Invgamma	0.10	2.00
σ_g	Government sp. shock	Invgamma	0.10	2.00
σ_I	Investment-specific shock	Invgamma	0.10	2.00
σ_r	Monetary policy shock	Invgamma	0.10	2.00
σ_p	Price mark-up shock	Invgamma	0.10	2.00
σ_w	Wage mark-up shock	Invgamma	0.10	2.00
σ_ν	Sunspot shock	Uniform[0,1]	0.50	0.29
ρ_a	Persistence technology	Beta	0.50	0.20
ρ_b	Persistence risk premium	Beta	0.50	0.20
ρ_g	Persistence government sp.	Beta	0.50	0.20
ρ_I	Persistence investment-specific	Beta	0.50	0.20
ρ_r	Persistence monetary policy	Beta	0.50	0.20
ρ_p	Persistence price mark-up	Beta	0.50	0.20
ρ_w	Persistence wage mark-up	Beta	0.50	0.20
μ_p	Mov. Avg. term, price mark-up	Beta	0.50	0.20
μ_w	Mov. Avg. term, wage mark-up	Beta	0.50	0.20
ρ_{ga}	$Cov(\sigma_a, \sigma_g)$	Normal	0.50	0.25
$\rho_{\nu p}$	$Corr(\sigma_\nu, \sigma_p)$	Uniform[-1,1]	0	0.57

Table K.2: Prior distributions for the exogenous processes of the model.

L Appendix IV.B

Figure L.1 plots the historical decomposition of the output gap for two alternative specifications. The panel at the top decomposes the output gap for the case of a failure to stabilize the economy, as shown in Section 5.1. The bottom panel reports the decomposition that results from the assumption of equilibrium uniqueness as conducted in SW. The two plots indicate minor differences and attribute the recessions of the late 1950s to demand shocks and the contractions of the early 1970s to a combination of mark-up and demand shocks. Also, non-fundamental disturbances had almost no effect on the observed fluctuations in the output gap. The similarity of the decomposition should not come as a surprise. Indeed, the analysis conducted in Section 5.2 shows that the transmission of the structural shocks on the output gap is roughly invariant to the of monetary policy stance, given that the differences in the magnitudes are due to the larger size of the estimated standard deviations of the shocks for the pre-Volcker period.

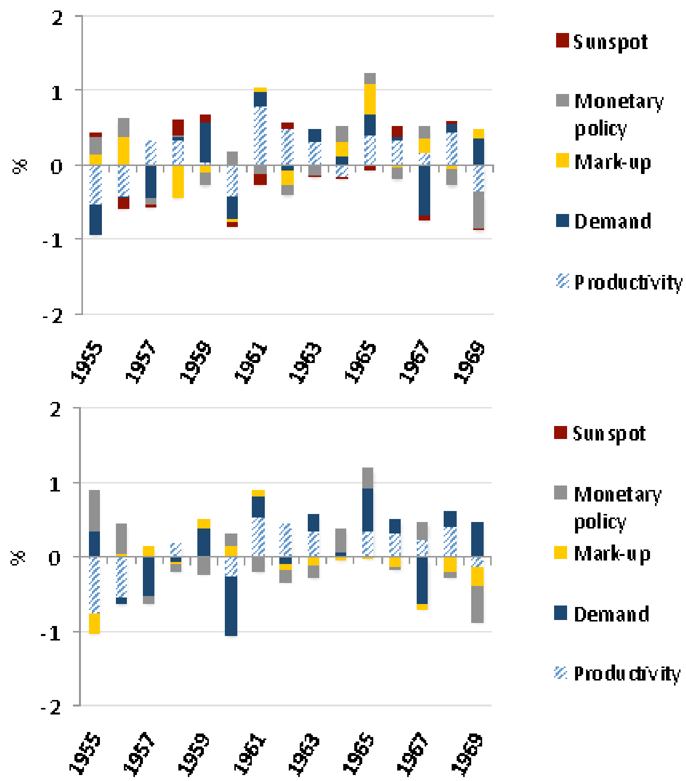

Figure L.1: Sample 1955-1969. Historical decomposition of the output gap under indeterminacy (top) and determinacy (bottom).

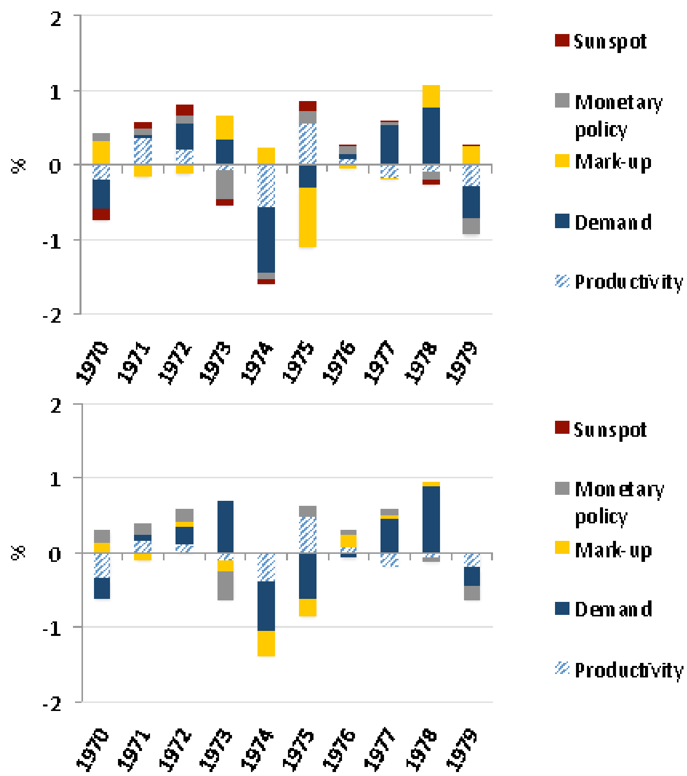

Figure L.2: Sample 1970-1979. Historical decomposition of the output gap under indeterminacy (top) and determinacy (bottom).

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