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# Stochastic Orders

 Springer

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## Preface

Stochastic orders and inequalities have been used during the last 40 years, at an accelerated rate, in many diverse areas of probability and statistics. Such areas include reliability theory, queuing theory, survival analysis, biology, economics, insurance, actuarial science, operations research, and management science. The purpose of this book is to collect in one place essentially all that is known about these orders up to the present. In addition, the book illustrates some of the usefulness and applicability of these stochastic orders.

This book is a major extension of the first six chapters in Shaked and Shanthikumar [515]. The idea that led us to write those six chapters arose as follows. In our own research in reliability theory and operations research we have been using, for years, several notions of stochastic orders. Often we would encounter a result that we could easily (or not so easily) prove, but we could not tell whether it was known or new. Even when we were sure that a result was known, we would not know right away where it could be found. Also, sometimes we would prove a result for the purpose of an application, only to realize later that a stronger result (stronger than what we needed) had already been derived elsewhere. We also often have had difficulties giving a reference for *one* source that contained everything about stochastic orders that we needed in a particular paper. In order to avoid such difficulties we wrote the first six chapters in Shaked and Shanthikumar [515].

Since 1994 the theory of stochastic orders has grown significantly. We think that now is the time to put in one place essentially all that is known about these orders. This book is the result of this effort.

The simplest way of comparing two distribution functions is by the comparison of the associated means. However, such a comparison is based on only two single numbers (the means), and therefore it is often not very informative. In addition to this, the means sometimes do not exist. In many instances in applications one has more detailed information, for the purpose of comparison of two distribution functions, than just the two means. Several orders of distribution functions, that take into account various forms of possible knowl-

edge about the two underlying distribution functions, are studied in Chapters 1 and 2.

When one wishes to compare two distribution functions that have the same mean (or that are centered about the same value), one is usually interested in the comparison of the dispersion of these distributions. The simplest way of doing it is by the comparison of the associated standard deviations. However, such a comparison, again, is based on only two single numbers, and therefore it is often not very informative. In addition to this, again, the standard deviations sometimes do not exist. Several orders of distribution functions, which take into account various forms of possible knowledge about the two underlying distribution functions (in addition to the fact that they are centered about the same value), are studied in Chapter 3. Orders that can be used for the joint comparison of both the location and the dispersion of distribution functions are studied in Chapters 4 and 5. The analogous orders for multivariate distribution functions are studied in Chapters 6 and 7.

When one is interested in the comparison of a sequence of distribution functions, associated with the random variables  $X_i$ ,  $i = 1, 2, \dots$ , then one can use, of course, any of the orders described in Chapters 1–7 for the purpose of comparing any two of these distributions. However, the parameter  $i$  may now introduce some patterns that connect all the underlying distributions. For example, suppose not only that the random variables  $X_i$ ,  $i = 1, 2, \dots$ , increase stochastically in  $i$ , but also that the increase is sharper for larger  $i$ 's. Then the sequence  $X_i$ ,  $i = 1, 2, \dots$ , is stochastically increasing in a convex sense. Such notions of stochastic convexity and concavity are studied in Chapter 8.

Notions of positive dependence of two random variables  $X_1$  and  $X_2$  have been introduced in the literature in an effort to mathematically describe the property that “large (respectively, small) values of  $X_1$  go together with large (respectively, small) values of  $X_2$ .” Many of these notions of positive dependence are defined by means of some comparison of the joint distribution of  $X_1$  and  $X_2$  with their distribution under the theoretical assumption that  $X_1$  and  $X_2$  are independent. Often such a comparison can be extended to general pairs of bivariate distributions with given marginals. This fact led researchers to introduce various notions of positive dependence orders. These orders are designed to compare the strength of the positive dependence of the two underlying bivariate distributions. Many of these orders can be further extended to comparisons of general multivariate distributions that have the same marginals. In Chapter 9 we describe these orders.

We have in mind a wide spectrum of readers and users of this book. On one hand, the text can be useful for those who are already familiar with many aspects of stochastic orders, but who are not aware of all the developments in this area. On the other hand, people who are not very familiar with stochastic orders, but who know something about them, can use this book for the purpose of studying or widening their knowledge and understanding of this important area.

We wish to thank Haijun Li, Asok K. Nanda, and Taizhong Hu for critical readings of several drafts of the manuscript. Their comments led to a substantial improvement in the presentation of some of the results in these chapters. We also thank Yigal Gerchak and Marco Scarsini for some illuminating suggestions. We thank our academic advisors John A. Buzacott (of J. G. S.) and Albert W. Marshall (of M. S.) who, years ago, introduced us to some aspects of the area of stochastic orders.

Tucson, Berkeley,  
August 16, 2006

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# Contents

<b>1</b>	<b>Univariate Stochastic Orders</b>	<b>3</b>
1.A	The Usual Stochastic Order	3
1.A.1	Definition and equivalent conditions	3
1.A.2	A characterization by construction on the same probability space	4
1.A.3	Closure properties	5
1.A.4	Further characterizations and properties	8
1.A.5	Some properties in reliability theory	15
1.B	The Hazard Rate Order	16
1.B.1	Definition and equivalent conditions	16
1.B.2	The relation between the hazard rate and the usual stochastic orders	18
1.B.3	Closure properties and some characterizations	18
1.B.4	Comparison of order statistics	31
1.B.5	Some properties in reliability theory	35
1.B.6	The reversed hazard order	36
1.C	The Likelihood Ratio Order	42
1.C.1	Definition	42
1.C.2	The relation between the likelihood ratio and the hazard and reversed hazard orders	43
1.C.3	Some properties and characterizations	44
1.C.4	Shifted likelihood ratio orders	66
1.D	The Convolution Order	70
1.E	Complements	71
<b>2</b>	<b>Mean Residual Life Orders</b>	<b>81</b>
2.A	The Mean Residual Life Order	81
2.A.1	Definition	81
2.A.2	The relation between the mean residual life and some other stochastic orders	83
2.A.3	Some closure properties	86

2.A.4	A property in reliability theory . . . . .	94
2.B	The Harmonic Mean Residual Life Order . . . . .	94
2.B.1	Definition . . . . .	94
2.B.2	The relation between the harmonic mean residual life and some other stochastic orders . . . . .	95
2.B.3	Some closure properties . . . . .	97
2.B.4	Properties in reliability theory . . . . .	105
2.C	Complements . . . . .	106
<b>3</b>	<b>Univariate Variability Orders</b> . . . . .	<b>109</b>
3.A	The Convex Order . . . . .	109
3.A.1	Definition and equivalent conditions . . . . .	109
3.A.2	Closure and other properties . . . . .	119
3.A.3	Conditions that lead to the convex order . . . . .	133
3.A.4	Some properties in reliability theory . . . . .	138
3.A.5	The $m$ -convex orders . . . . .	139
3.B	The Dispersive Order . . . . .	146
3.B.1	Definition and equivalent conditions . . . . .	146
3.B.2	Properties . . . . .	151
3.C	The Excess Wealth Order . . . . .	163
3.C.1	Motivation and definition . . . . .	163
3.C.2	Properties . . . . .	165
3.D	The Peakedness Order . . . . .	171
3.D.1	Definition . . . . .	171
3.D.2	Some properties . . . . .	172
3.E	Complements . . . . .	174
<b>4</b>	<b>Univariate Monotone Convex and Related Orders</b> . . . . .	<b>181</b>
4.A	The Monotone Convex and Monotone Concave Orders . . . . .	181
4.A.1	Definitions and equivalent conditions . . . . .	181
4.A.2	Closure properties and some characterizations . . . . .	185
4.A.3	Conditions that lead to the increasing convex and increasing concave orders . . . . .	193
4.A.4	Further properties . . . . .	197
4.A.5	Some properties in reliability theory . . . . .	203
4.A.6	The starshaped order . . . . .	204
4.A.7	Some related orders . . . . .	206
4.B	Transform Orders: Convex, Star, and Superadditive Orders . . . . .	213
4.B.1	Definitions . . . . .	213
4.B.2	Some properties . . . . .	214
4.B.3	Some related orders . . . . .	221
4.C	Complements . . . . .	227

**5 The Laplace Transform and Related Orders** ..... 233

5.A The Laplace Transform Order ..... 233

5.A.1 Definitions and equivalent conditions ..... 233

5.A.2 Closure and other properties ..... 235

5.B Orders Based on Ratios of Laplace Transforms ..... 245

5.B.1 Definitions and equivalent conditions ..... 245

5.B.2 Closure properties ..... 246

5.B.3 Relationship to other stochastic orders ..... 249

5.C Some Related Orders ..... 252

5.C.1 The factorial moments order ..... 252

5.C.2 The moments order ..... 255

5.C.3 The moment generating function order ..... 260

5.D Complements ..... 261

**6 Multivariate Stochastic Orders** ..... 265

6.A Notations and Preliminaries ..... 265

6.B The Usual Multivariate Stochastic Order ..... 266

6.B.1 Definition and equivalent conditions ..... 266

6.B.2 A characterization by construction on the same probability space ..... 266

6.B.3 Conditions that lead to the multivariate usual stochastic order ..... 267

6.B.4 Closure properties ..... 273

6.B.5 Further properties ..... 275

6.B.6 A property in reliability theory ..... 279

6.B.7 Stochastic ordering of stochastic processes ..... 280

6.C The Cumulative Hazard Order ..... 286

6.C.1 Definition ..... 286

6.C.2 The relationship between the cumulative hazard order and the usual multivariate stochastic order ..... 288

6.D Multivariate Hazard Rate Orders ..... 290

6.D.1 Definitions and basic properties ..... 290

6.D.2 Preservation properties ..... 292

6.D.3 The dynamic multivariate hazard rate order ..... 294

6.E The Multivariate Likelihood Ratio Order ..... 298

6.E.1 Definition ..... 298

6.E.2 Some properties ..... 298

6.E.3 A property in reliability theory ..... 304

6.F The Multivariate Mean Residual Life Order ..... 305

6.F.1 Definition ..... 305

6.F.2 The relation between the multivariate mean residual life and the dynamic multivariate hazard rate orders ... 306

6.F.3 A property in reliability theory ..... 307

6.G Other Multivariate Stochastic Orders ..... 307

6.G.1 The orthant orders ..... 307

6.G.2	The scaled order statistics orders	314
6.H	Complements	317
<b>7</b>	<b>Multivariate Variability and Related Orders</b>	<b>323</b>
7.A	The Monotone Convex and Monotone Concave Orders	323
7.A.1	Definitions	323
7.A.2	Closure properties	326
7.A.3	Further properties	328
7.A.4	Convex and concave ordering of stochastic processes	330
7.A.5	The $(m_1, m_2)$ -icx orders	331
7.A.6	The symmetric convex order	332
7.A.7	The componentwise convex order	333
7.A.8	The directional convex and concave orders	335
7.A.9	The orthant convex and concave orders	339
7.B	Multivariate Dispersion Orders	342
7.B.1	A strong multivariate dispersion order	342
7.B.2	A weak multivariate dispersion order	344
7.B.3	Dispersive orders based on constructions	346
7.C	Multivariate Transform Orders: Convex, Star, and Superadditive Orders	348
7.D	The Multivariate Laplace Transform and Related Orders	349
7.D.1	The multivariate Laplace transform order	349
7.D.2	The multivariate factorial moments order	352
7.D.3	The multivariate moments order	353
7.E	Complements	354
<b>8</b>	<b>Stochastic Convexity and Concavity</b>	<b>357</b>
8.A	Regular Stochastic Convexity	357
8.A.1	Definitions	358
8.A.2	Closure properties	362
8.A.3	Stochastic $m$ -convexity	365
8.B	Sample Path Convexity	367
8.B.1	Definitions	367
8.B.2	Closure properties	370
8.C	Convexity in the Usual Stochastic Order	374
8.C.1	Definitions	374
8.C.2	Closure properties	376
8.D	Strong Stochastic Convexity	377
8.D.1	Definitions	377
8.D.2	Closure properties	380
8.E	Stochastic Directional Convexity	381
8.E.1	Definitions	381
8.E.2	Closure properties	382
8.F	Complements	384



**9 Positive Dependence Orders** . . . . . 387

9.A The PQD and the Supermodular Orders . . . . . 387

    9.A.1 Definition and basic properties: The bivariate case . . . . . 387

    9.A.2 Closure properties . . . . . 390

    9.A.3 The multivariate case . . . . . 392

    9.A.4 The supermodular order . . . . . 395

9.B The Orthant Ratio Orders . . . . . 404

    9.B.1 The (weak) orthant ratio orders . . . . . 404

    9.B.2 The strong orthant ratio orders . . . . . 407

9.C The LTD, RTI, and PRD Orders . . . . . 408

9.D The PLRD Order . . . . . 414

9.E Association Orders . . . . . 417

9.F The PDD Order . . . . . 420

9.G Ordering Exchangeable Distributions . . . . . 423

9.H Complements . . . . . 426

**References** . . . . . 431

**Author Index** . . . . . 459

**Subject Index** . . . . . 467