

Foundations of Quantitative Finance

Book I: Measure Spaces and Measurable Functions

Robert R. Reitano
Brandeis International Business School
Waltham, MA 02454



CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

Contents

Preface.....	xi
Author Bio.....	xiii
Introduction.....	xv
1. The Notion of Measure 0.....	1
1.1 Riemann Integrals.....	3
1.2 Lebesgue Integrals.....	5
2. Lebesgue Measure on \mathbb{R}.....	11
2.1 Sigma Algebras and Borel Sets.....	11
2.2 Definition of a Lebesgue Measure.....	20
2.3 Is There a Lebesgue Measure on $\sigma(P(\mathbb{R}))$?.....	23
2.4 Lebesgue Measurable Sets: $\mathcal{M}_L(\mathbb{R}) \subsetneq \sigma(P(\mathbb{R}))$	31
2.5 Calculating Lebesgue Measures.....	38
2.6 Approximating Lebesgue Measurable Sets.....	39
2.7 Properties of Lebesgue Measure.....	41
2.7.1 Regularity.....	41
2.7.2 Continuity.....	43
2.8 Discussion of $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{M}_L(\mathbb{R})$	45
3. Measurable Functions.....	47
3.1 Extended Real-Valued Functions.....	47
3.2 Equivalent Definitions of Measurability.....	48
3.3 Examples of Measurable Functions.....	52
3.3.1 Continuous Functions.....	53
3.3.2 Characteristic or Indicator Functions.....	59
3.3.3 A Nonmeasurable Function.....	61
3.4 Properties of Measurable Functions.....	62
3.4.1 Elementary Function Combinations.....	65
3.4.2 Function Sequences.....	68
3.5 Approximating Measurable Functions.....	74
3.6 Distribution Functions.....	79
4. Littlewood's Three Principles.....	87
4.1 Measurable Sets.....	87
4.2 Sequences of Measurable Functions.....	89
4.3 Measurable Functions.....	91
5. Borel Measures on \mathbb{R}.....	93
5.1 Functions Induced by Measures.....	95
5.2 Measures from Distribution Functions.....	98
5.2.1 F -Length to a Measure on an Algebra.....	99

5.2.2	To a Borel Measure.....	104
5.3	Consistency of Borel Constructions.....	111
5.4	Approximating Borel Measurable Sets.....	113
5.5	Properties of Borel Measures.....	116
5.5.1	Continuity.....	116
5.5.2	Regularity.....	116
5.6	Differentiable $F(x)$	120
6.	Measures by Extension.....	125
6.1	Recap of Lebesgue and Borel Constructions.....	125
6.2	Extension Theorems.....	126
6.2.1	From Outer Measure to Complete Measure.....	127
6.2.2	Measure on an Algebra to a Complete Measure.....	130
6.2.3	Approximating Carathéodory Measurable Sets.....	132
6.2.4	Pre-Measure on Semi-Algebra to Measure on Algebra.....	134
6.2.5	Uniqueness of Extensions 1.....	140
6.3	Summary of Construction Process.....	143
6.4	Approaches to Countable Additivity.....	145
6.4.1	Countable Additivity on a Semi-Algebra.....	145
6.4.2	Countable Additivity on an Algebra.....	147
6.5	Completion of a Measure Space.....	149
6.5.1	Uniqueness of Extensions 2.....	151
7.	Finite Products of Measure Spaces.....	157
7.1	Product Space Semi-Algebras.....	157
7.2	Properties of the Semi-Algebra \mathcal{A}'	161
7.3	Measure on the Algebra \mathcal{A}	164
7.3.1	Finite Additivity on the Semi-Algebra \mathcal{A}'	166
7.3.2	Countable Additivity on the Algebra \mathcal{A} for σ -Finite Spaces.....	173
7.4	Extension to a Measure on the Product Space.....	178
7.5	Well-Definedness of Product Measure Spaces.....	179
7.6	Lebesgue and Borel Product Spaces.....	184
8.	Borel Measures on \mathbb{R}^n.....	187
8.1	Rectangle Collections that Generate $\mathcal{B}(\mathbb{R}^n)$	187
8.2	Borel Measures and Induced Functions.....	189
8.2.1	Functions Induced by Finite Borel Measures.....	190
8.2.2	Functions Induced by General Borel Measures.....	196
8.2.3	Borel Measures Induced by Functions.....	202
8.3	Properties of Borel Measures on \mathbb{R}^n	210
8.3.1	Uniqueness and Consistency.....	210
8.3.2	Approximating Borel Measurable Sets.....	213
8.3.3	Continuity and Regularity.....	214

9. Infinite Product Spaces	219
9.1 Naive Attempt at a First Step	219
9.2 Semi-Algebra \mathcal{A}'	221
9.3 Finite Additivity of $\mu_{\mathcal{A}}$ on \mathcal{A}	225
9.4 Free Countable Additivity on Finite Spaces.....	227
9.5 Countable Additivity on \mathcal{A}^+	230
9.5.1 Outline of Proof and Need for \mathcal{A}^+	231
9.5.2 Algebra \mathcal{A}^+ and Finite Additivity of $\mu_{\mathcal{A}}$	234
9.5.3 Countable Additivity of $\mu_{\mathcal{A}}$ on \mathcal{A}^+	240
9.6 Extension to a Probability Measure on $\mathbb{R}^{\mathbb{N}}$	243
9.7 Probability of General Rectangles	246
References	249
Index	253