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An Introduction to Mathematical Finance with Applications

Understanding and Building Financial Intuition

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