Donald G. Saari

Mathematics of Finance

An Intuitive Introduction

Springer

Contents

1 Preliminaries via Gambles 1
1.1 A Football Game 1
1.1.1 Removing Uncertainty 2
1.1.2 Computations 3
1.1.3 Profit Curves 4
1.1.4 Fixed Income 4
1.1.5 Arbitrage 5
1.1.6 Hedging Without Arbitrage 6
1.1.7 A Cautionary Word of Interpretation 7
1.2 Expected Value and Variance 9
1.2.1 Probability and PDF 9
1.2.2 Random Variable 14
1.2.3 Expected Value 14
1.2.4 Variance 15
1.2.5 Standard Form 18
1.3 Fair Bets and Ensuring Profits 18
1.3.1 Fair Bet 19
1.3.2 Making Money 20
1.3.3 Horse Racing 21
1.4 Exercises 24

2 Options 29
2.1 Calls 29
2.1.1 A Long Call 30
2.1.2 Going Short 33
2.1.3 Hedging 34
2.2 Puts 35
2.2.1 A Long Put 35
2.2.2 Short Put 37
2.2.3 Some Jargon 37

2.3 Hedging ... 38
2.3.1 Straddle ... 38
2.3.2 Designing Portfolios ... 39
2.4 Put–Call Parity ... 42
2.4.1 Present Value of Money ... 43
2.4.2 Security ... 44
2.5 Information Gained ... 47
2.5.1 Our Friend: Arbitrage ... 48
2.5.2 Properties of Puts and Calls ... 49
2.6 Exercises ... 50

3 Modeling ... 55
3.1 Assumptions and Modeling ... 55
3.1.1 Taylor Series ... 56
3.1.2 More Variables ... 59
3.1.3 Back to Modeling Approximations ... 60
3.2 Efficient Market Hypothesis ... 60
3.2.1 Modeling ... 62
3.2.2 Random Variables ... 63
3.2.3 Back to Finance ... 65
3.2.4 Random Effects ... 67
3.3 Interpretation ... 68
3.3.1 Probability Distribution ... 69
3.4 Exercises ... 69

4 Some Probability ... 71
4.1 Review of Some Probability ... 71
4.1.1 Review of Chain Rule ... 74
4.1.2 Finding New PDFs ... 76
4.2 Itô's Lemma ... 77
4.3 Application ... 79
4.3.1 PDF for S(t) ... 81
4.3.2 Lognormal Distribution ... 82
4.4 Exercises ... 84

5 The Black–Scholes Equation ... 89
5.1 Black–Scholes Equation ... 90
5.2 Boundary Conditions ... 93
5.2.1 Heat Equation ... 93
5.2.2 Black–Scholes Boundary Conditions ... 94
5.3 Conversion to the Heat Equation ... 97
5.3.1 A Quick Tutorial in Differential Equations ... 97
5.3.2 Eliminating the Variable Coefficients ... 101
5.4 Intuition ... 102
5.5 Exercises ... 103

6 Solutions of Black–Scholes 105
6.1 The Heat Equation and $C_E(S,t)$ Solution 105
6.2 Source of $C_E(S,t)$ Terms 106
6.3 Interpretation 107
6.4 Exercises 109

7 Partial Information: The Greeks 111
7.1 The $P_E(S,t)$ Solution 111
7.2 Here Come the Greeks! 112
7.2.1 The Hedge Ratio δ Term 112
7.2.2 Changing δ; the Gamma Greek Γ 116
7.2.3 The Fake Greek—vega ν 117
7.2.4 More Members of the Greek Party 118
7.3 Exercises 120

8 Sketching and the American Options 121
8.1 Using δ_C and δ_P to Sketch $C_E(S,t)$ and $P_E(S,t)$ 121
8.1.1 Sketching $C_E(S,t)$ 121
8.1.2 Sketching $P_E(S,t)$ 123
8.1.3 Comparing Curves 124
8.2 Arbitrage and the American Option 124
8.2.1 Simple Geometry for Puts 125
8.2.2 Arbitrage with a Call 126
8.2.3 New Rules; the American Option 129
8.3 Exercises 130

9 Embellishments 131
9.1 Bonds 131
9.2 Dividends, or Other Embellishments 133
9.2.1 A New Problem 134
9.2.2 Here Comes the Solution 135
9.3 Numerical Integration 136
9.3.1 The Heat Equation 136
9.4 What Is Next? 139
9.5 Exercises 139

References 141

Index 143