Stephen Abbott

Understanding Analysis

Second Edition

Springer

Contents

1 The Real Numbers **1**
- 1.1 Discussion: The Irrationality of $\sqrt{2}$ . . . 1
- 1.2 Some Preliminaries . . . 5
- 1.3 The Axiom of Completeness . . . 14
- 1.4 Consequences of Completeness . . . 20
- 1.5 Cardinality . . . 25
- 1.6 Cantor's Theorem . . . 32
- 1.7 Epilogue . . . 36

2 Sequences and Series **39**
- 2.1 Discussion: Rearrangements of Infinite Series . . . 39
- 2.2 The Limit of a Sequence . . . 42
- 2.3 The Algebraic and Order Limit Theorems . . . 49
- 2.4 The Monotone Convergence Theorem and a First Look at Infinite Series . . . 56
- 2.5 Subsequences and the Bolzano–Weierstrass Theorem . . . 62
- 2.6 The Cauchy Criterion . . . 66
- 2.7 Properties of Infinite Series . . . 71
- 2.8 Double Summations and Products of Infinite Series . . . 79
- 2.9 Epilogue . . . 83

3 Basic Topology of R **85**
- 3.1 Discussion: The Cantor Set . . . 85
- 3.2 Open and Closed Sets . . . 88
- 3.3 Compact Sets . . . 96
- 3.4 Perfect Sets and Connected Sets . . . 102
- 3.5 Baire's Theorem . . . 106
- 3.6 Epilogue . . . 109

4 Functional Limits and Continuity **111**
- 4.1 Discussion: Examples of Dirichlet and Thomae . . . 111
- 4.2 Functional Limits . . . 115

4.3 Continuous Functions . . . 122
4.4 Continuous Functions on Compact Sets . . . 129
4.5 The Intermediate Value Theorem . . . 136
4.6 Sets of Discontinuity . . . 141
4.7 Epilogue . . . 144

5 The Derivative **145**
5.1 Discussion: Are Derivatives Continuous? . . . 145
5.2 Derivatives and the Intermediate Value Property . . . 148
5.3 The Mean Value Theorems . . . 155
5.4 A Continuous Nowhere-Differentiable Function . . . 162
5.5 Epilogue . . . 166

6 Sequences and Series of Functions **169**
6.1 Discussion: The Power of Power Series . . . 169
6.2 Uniform Convergence of a Sequence of Functions . . . 173
6.3 Uniform Convergence and Differentiation . . . 184
6.4 Series of Functions . . . 188
6.5 Power Series . . . 191
6.6 Taylor Series . . . 197
6.7 The Weierstrass Approximation Theorem . . . 205
6.8 Epilogue . . . 211

7 The Riemann Integral **215**
7.1 Discussion: How Should Integration be Defined? . . . 215
7.2 The Definition of the Riemann Integral . . . 218
7.3 Integrating Functions with Discontinuities . . . 224
7.4 Properties of the Integral . . . 228
7.5 The Fundamental Theorem of Calculus . . . 234
7.6 Lebesgue's Criterion for Riemann Integrability . . . 238
7.7 Epilogue . . . 246

8 Additional Topics **249**
8.1 The Generalized Riemann Integral . . . 249
8.2 Metric Spaces and the Baire Category Theorem . . . 258
8.3 Euler's Sum . . . 264
8.4 Inventing the Factorial Function . . . 270
8.5 Fourier Series . . . 281
8.6 A Construction of **R** From **Q** . . . 297

Bibliography **305**

Index **307**