

David Borthwick

Introduction to Partial Differential Equations



Springer

Contents

1	Introduction	1
1.1	Partial Differential Equations	1
1.2	Example: d'Alembert's Wave Equation	2
1.3	Types of Equations	3
1.4	Well Posed Problems	5
1.5	Approaches	6
2	Preliminaries	9
2.1	Real Numbers	9
2.2	Complex Numbers	10
2.3	Domains in \mathbb{R}^n	12
2.4	Differentiability	13
2.5	Ordinary Differential Equations	15
2.6	Vector Calculus	18
2.7	Exercises	23
3	Conservation Equations and Characteristics	25
3.1	Model Problem: Oxygen in the Bloodstream	25
3.2	Lagrangian Derivative and Characteristics	27
3.3	Higher-Dimensional Equations	32
3.4	Quasilinear Equations	35
3.5	Exercises	41
4	The Wave Equation	45
4.1	Model Problem: Vibrating String	45
4.2	Characteristics	47
4.3	Boundary Problems	51
4.4	Forcing Terms	53
4.5	Model Problem: Acoustic Waves	59
4.6	Integral Solution Formulas	61
4.7	Energy and Uniqueness	67
4.8	Exercises	69

5 Separation of Variables	75
5.1 Model Problem: Overtones	76
5.2 Helmholtz Equation	76
5.3 Circular Symmetry	81
5.4 Spherical Symmetry	87
5.5 Exercises	93
6 The Heat Equation	97
6.1 Model Problem: Heat Flow in a Metal Rod	97
6.2 Scale-Invariant Solution	101
6.3 Integral Solution Formula	103
6.4 Inhomogeneous Problem	107
6.5 Exercises	109
7 Function Spaces	111
7.1 Inner Products and Norms	111
7.2 Lebesgue Integration	114
7.3 L^p Spaces	116
7.4 Convergence and Completeness	119
7.5 Orthonormal Bases	122
7.6 Self-adjointness	125
7.7 Exercises	128
8 Fourier Series	131
8.1 Series Solution of the Heat Equation	131
8.2 Periodic Fourier Series	134
8.3 Pointwise Convergence	138
8.4 Uniform Convergence	141
8.5 Convergence in L^2	143
8.6 Regularity and Fourier Coefficients	145
8.7 Exercises	151
9 Maximum Principles	155
9.1 Model Problem: The Laplace Equation	155
9.2 Mean Value Formula	161
9.3 Strong Principle for Subharmonic Functions	165
9.4 Weak Principle for Elliptic Equations	167
9.5 Application to the Heat Equation	170
9.6 Exercises	174
10 Weak Solutions	177
10.1 Test Functions and Weak Derivatives	177
10.2 Weak Solutions of Continuity Equations	181
10.3 Sobolev Spaces	187
10.4 Sobolev Regularity	190

Contents	xiii
10.5 Weak Formulation of Elliptic Equations	194
10.6 Weak Formulation of Evolution Equations	196
10.7 Exercises	202
11 Variational Methods	205
11.1 Model Problem: The Poisson Equation	206
11.2 Dirichlet's Principle	207
11.3 Coercivity and Existence of a Minimum	208
11.4 Elliptic Regularity	214
11.5 Eigenvalues by Minimization	217
11.6 Sequential Compactness	224
11.7 Estimation of Eigenvalues	227
11.8 Euler-Lagrange Equations	234
11.9 Exercises	237
12 Distributions	239
12.1 Model Problem: Coulomb's Law	239
12.2 The Space of Distributions	242
12.3 Distributional Derivatives	245
12.4 Fundamental Solutions	248
12.5 Green's Functions	252
12.6 Time-Dependent Fundamental Solutions	257
12.7 Exercises	259
13 The Fourier Transform	261
13.1 Fourier Transform	261
13.2 Tempered Distributions	267
13.3 The Wave Kernel	271
13.4 The Heat Kernel	273
13.5 Exercises	273
Erratum to: Introduction to Partial Differential Equations	E1
Appendix A: Analysis Foundations	277
References	281
Index	283